Malaysian Journal of Mathematical Sciences 19(2): 613–635 (2025)
https://doi.org/10.47836/mjms.19.2.13

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

On Examining Metric Dimension Through Edge Contraction in Certain Families of Graphs

Saleem, H[1], Husin, M. N.*[2], Ali, S.[3], and Hameed, M. S.[1]

[1]*Institute of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, 64200, Pakistan*
[2]*Special Interest Group on Modeling and Data Analytics, Faculty of Computer Science and Mathematical, Universiti Malaysia Terengganu, Kuala Nerus 21030, Terengganu, Malaysia*
[3]*Department of Mathematics, The Islamia University of Bahawalpur, Rahim Yar Khan Campus, 64200, Pakistan*

E-mail:nazri.husin@umt.edu.my
**Corresponding author*

Received: 10 May 2024
Accepted: 26 December 2024

Abstract

In graph theory, the Metric Dimension (MD) is an elementary metric that affords evidence nearly the essential selves of graphs. We reconnoiter the MD in the venue of edge-contracted regular graphs in this paper, with exceptional devotion to the Antiprism, Petersen, and Harary graphs. Our effort creates a vital bond between antiprism and its edge-contracted counterpart: we give a scheme to regulate the MD of the edge-contracted graph, given the MD of the novel graph. We likewise inspect how edge contraction affects regular graphs' MDs, providing insight into how this operation deviations both the MD and the underlying graph topology. By providing supportive means for the exploration and alteration of regular graphs in a variety of real-world circumstances, our investigation elucidates these belongings and improves the field of graph theory.

Keywords: edge contraction; metric dimension; antiprism graph; Petersen graph; Harary graph.

1 Introduction

A *Resolving Set* (RS) is a subset of vertices in a graph G that exclusively governs the extent between every duo of vertices. Any duo of vertices u and v in V has a vertex r in the RS $\Re$ with an altered distance between v and r than u and r. The *MD* of a graph G is the cardinality of its smallest RS, which is signified by the cipher $dim(G)$. Identifying a RS or calculating a graph's MD are difficult computer problems that are frequently NP-hard [22]. Contingent on the grid's inimitable potentials, abundant strategies, and heuristics have been anticipated to ascertain accurate responses or imprecise them.

In graph theory, an Edge Contraction (EC) is the merger of dual vertices along an edge to form an innovative graph with one less vertex. Many graph operations and procedures can advantage from EC. It can be used, for instance, in graph partitioning, network flow calculations, graph simplification, and graph isomorphism te]sting algorithms. But it's crucial to remember that EC modifies the graph's structure, possibly affecting its diameter, clustering coefficient, and other structural elements.

The imperative task of pinpointing an intruder within a network, devoid of ambiguity, served as the impetus for Slater's conceptualization of the MD of a graph, as delineated in [18, 20]. Independently, Harary and Melter delved into this concept in their own right, as documented in [11]. This cardinal property finds diverse applications across various domains, such as; resolvability in graphs [7], and metric bases in digital geometry are discussed in [14]. Explorations into the MD have extended beyond its theoretical roots to encompass analysis of graph operations and structured outputs.

Notably, investigations have probed the MD of graph transformations such as the line graph [9], and comarsion of MD with different families of graphs are computed [9]. The Cartesian product of graphs [6] and their MD properties are discussed [16]. The product of corona graphs [12] and the MD basis properties are computed [23]. The product of joint graphs [15], MD base,d and local fractional based are developed [21]. The product of lexicographic graphs [17], and the product of hierarchical graphs [8] are discussed in detail. The study of local fractional metric dimension of rotationally symmetrical polygonal graphs [3].

The MDs and their extensions for the generalized perimantanes diamondoid structure, demonstrating how each parameter varies with the parameter n, or the number of copies, and is dependent on the copies of the original or base perimantanes diamondoid structure has been study in [5]. The classification of trees and unicyclic graphs with mixed dimension three, and further to determine the specific requirements that must be met for a graph to have mixed MD 3 [4]. Alamer et al. [2] established precise limits on the local fractional MD of several modified prism network types. It is also demonstrated that the local fractional MD of these networks stays constrained as their order approaches infinity.

The boundedness properties for local fractional MD over an algebraic structure graph has been studied in [1]. The modified symmetric division deg index is studied in [10]. Recent research on the MD of carbon nanotube Y-junctions [19] and supramolecular networks has made a remarkable contribution to the field. The primary findings of this study include MD of edge contracted families of graphs; namely Antiprism, Petersen, and Harary graphs. In Section 2-4, we compute the metric delusion of edge contracted Antiprism, edge contracted Petersen, and edge contracted Harary graphs respectively. In the last section we add concluding remarks regarding outcomes of the study.

The motive behind this effort is fantastic. The MD of a graph gauges the graph's endurance to reliably recognize the precise positions of vertices via distances to a particular class of vertices known as a RS. This notion has tangible uses in disciplines such as network navigation, combinatorial optimization, and biology, where effectively utilized differentiation between distinct vertices is critical. Considerate the MD relief scholars to ripen appliances for effectually tracing and monitoring evidence intimate a building.

EC, a scheme that unifies dual vertices coupled by an edge into a single vertex while confiscating self-loops, amends the distances among vertices, hence prompting the MD. Inspecting how MD develops throughout EC can give supplementary evidence on how essential interpretations influence a graph's resolving belongings. This is particularly expedient when large networks are exemplified by simpler, contracted models, as it permits extrapolations about the resolving proficiencies of these simplified graphs. Families including Antiprism, Petersen, and Harary graphs deliver inimitable sameness and regularity, making them supreme for reviewing the properties of EC. For example: Antiprism graphs have a symmetrical, cyclic shape, which sorts distance features simple to scrutinize and envisage. Discerning fluctuations in MD due to EC in these topologies delivers solid confirmation for appreciating how unvarying, cyclical graphs perform when abridged.

Petersen Graphs are well-studied due to their unique non-trivial assets, such as being both symmetric and non-planar, making them a perfect example for revising the intricacies of MD in complex graph topologies. Harary graphs are known for their high connectivity, which elasticities intuition into how contraction distresses distance metrics in heavily attached schemes. This fashions a substance for smearing the verdicts to other high-connectivity networks. Understanding MD fluctuations in contracted graphs is essential for large-scale networks where full network perseverance is computationally lavish. For example, in telecommunications networks, contracted models can reproduce large-scale networks while upholding crucial possessions for tracing and directing. Biological and chemical structures: in molecular and protein structure studies, contracted graph models are reduced approximations of complicated interaction networks, with MD insights helping to pinpoint crucial sites or connections.

Robotic navigation and surveillance: edge contraction in simplified pathfinding graphs can improve navigation by lowering complexity while keeping critical distance information. Studying EC and its impact on MD within certain graph families not only improves our understanding of these structures, but also contributes to larger theoretical insights into how MD operates across different transformation types. This might lead to the development of generalizable patterns or formulae for predicting MD outcomes across different graph alterations.

The novelty and originality of reviewing MD via EC for definite graph clans branch from influential how essential easy styles mark exceptional vertex identification, expressly in symmetrical or vastly linked graphs, which delivers both hypothetical visions and everyday solicitations for proficiently resolving and streamlining complex networks. While the MD is well known as a graph invariant, less attention has been fanatical to its performance under EC. The mainstream of MD inquiry attentions on stagnant structures or slighter conversions like vertex or edge elimination, whereas EC causes unique essential changes that have principally gone unfamiliar.

Real-world links (e.g., community, genetic, or transportation networks) are frequently amended to diminish computational complications. Your discoveries can give supervision for resembling MD in slim varieties of these linkages, possibly impacting productions that trust peak conversion or firmness performances. Integrating relevant invariants such as diameter, belonging, and chromatic number into your investigation or subsequent investigations can considerably increase our understanding of EC in certain graph families.

Here's how you could start it: in addition to MD, graph invariants like as dimension, connectedness, and chromatic number offer substantial insights when witnessed via the perspective of EC. Recognizing whether EC impacting specific features in graph families such as Antiprism, Petersen, and Harary graphs can aid in the identification of wider trends and connections in graph theory. This strategy has the capability to link MD to other invariants, so boosting intellectual understanding and applications for simplified network simulation and rapid graph feature assessment. By specifying these invariants, you broaden the extent of what you are investigating, perhaps leading to a comprehensive framework for predicting how distinct graph properties evolve under structural changes like EC.

2 Metric Dimension of Edge Contracted Antiprism Graph

The antiprism $A_{\eth_\triangleright}$ is a 4-regular graph. Antiprism $A_{\eth_\triangleright}$ consists of an outer cycle $u_1, u_2, \ldots, u_{\eth_\triangleright}$ and an inner cycle $v_1, v_2, \ldots, v_{\eth_\triangleright}$ and a set of $\eth_\triangleright$ spokes $v_{\daleth_\triangleright}u_{\daleth_\triangleright}$ and $v_{\daleth_\triangleright+1}u_{\daleth_\triangleright}$, $\daleth_\triangleright = 1, 2, \ldots, \eth_\triangleright$ with indices taken modulo $\eth_\triangleright$. $|V(A_{\eth_\triangleright})| = 2\eth_\triangleright$ and $E(A_{\eth_\triangleright}) = 4\eth_\triangleright$. Once more, selecting landmarks wisely is crucial.

Theorem 2.1. *Let $A_{\eth_\triangleright}$ with $\eth_\triangleright \geqslant 3$ be an Antiprism graph and $A_{\eth_\triangleright}.e$ be the outer edge contracted Antiprism graph then; $dim(A_{\eth_\triangleright}.e) = dim(A_{\eth_\triangleright})$ for $\eth_\triangleright \geqslant 3$.*

Proof. Let $\eth_\triangleright = 2\wp_\triangleright$ and $\eth_\triangleright = 2\wp_\triangleright + 1$ for $\eth_\triangleright$, even and odd, $\wp_\triangleright \in \mathbb{Z}^+$ with $\wp_\triangleright \geq 3$. Resolving set is $\{u_1, u_2, v_3\}$ for $\eth_\triangleright = 3$, and $\{u_1, u_2, u_3\}$ is a RS for $\eth_\triangleright = 4, 5$. Now for $\eth_\triangleright \geqslant 6$, we will show that RS for $V(A_{\eth_\triangleright}.e)$ is $\{u_1, u_3, u_{\wp_\triangleright+1}\}$. Firstly, we give vertices representation in $A_{\eth_\triangleright}.e$ as regards $\{u_1, u_3\}$. We can see that, $r(u_2|\{u_1, u_3\}) = (1, 1)$ and $r(u_1|\{u_1, u_3\}) = (0, 2)$ and $r(u_3|\{u_1, u_3\}) = (2, 0)$. Typically, the outer cycle vertices are represented as,

Case 1: If $\eth_\triangleright$ is odd and $\wp_\triangleright = 3$, then,

$$r(u_{\daleth_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\daleth_\triangleright - 1, \daleth_\triangleright - 3); & 4 \leq \daleth_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \daleth_\triangleright, \daleth_\triangleright - 3); & \wp_\triangleright + 2 \leq \daleth_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 2: If $\eth_\triangleright$ is odd and $\wp_\triangleright \geqslant 4$, then,

$$r(u_{\daleth_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\daleth_\triangleright - 1, \daleth_\triangleright - 3); & 4 \leq \daleth_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \daleth_\triangleright, \daleth_\triangleright - 3); & \wp_\triangleright + 2 \leq \daleth_\triangleright \leq \wp_\triangleright + 3, \\ (\eth_\triangleright - \daleth_\triangleright, \eth_\triangleright - \daleth_\triangleright + 2); & \wp_\triangleright + 4 \leq \daleth_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 3: For even $\eth_\triangleright$ and for $\wp_\triangleright = 3$,

$$r(u_{\daleth_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\daleth_\triangleright - 2, \daleth_\triangleright - 3); & 4 \leq \daleth_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \daleth_\triangleright, \daleth_\triangleright - 3); & \wp_\triangleright + 2 \leq \daleth_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 4: Even $\eth_\triangleright$ and for $\wp_\triangleright = 4$,

$$r(u_{\daleth_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\daleth_\triangleright - 1, \daleth_\triangleright - 3); & 4 \leq \daleth_\triangleright \leq \wp_\triangleright, \\ (\daleth_\triangleright - 2, \daleth_\triangleright - 3); & \daleth_\triangleright = \wp_\triangleright + 1, \\ (\eth_\triangleright - \daleth_\triangleright, \daleth_\triangleright - 3); & \daleth_\triangleright = \wp_\triangleright + 2, \\ (\eth_\triangleright - \daleth_\triangleright, \daleth_\triangleright - 4); & \wp_\triangleright + 3 \leq \daleth_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 5: If $\eth_{\triangleright}$ is even and for $\wp_{\triangleright} \geq 5$,

$$r(u_{\gimel_{\triangleright}}|\{u_1,u_3\}) = \begin{cases} (\gimel_{\triangleright}-1,\gimel_{\triangleright}-3); & 4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright}, \\ (\gimel_{\triangleright}-2,\gimel_{\triangleright}-3); & \gimel_{\triangleright} = \wp_{\triangleright}+1, \\ (\eth_{\triangleright}-\gimel_{\triangleright},\gimel_{\triangleright}-3); & \gimel_{\triangleright} = \wp_{\triangleright}+2, \\ (\eth_{\triangleright}-\gimel_{\triangleright},\gimel_{\triangleright}-4); & \gimel_{\triangleright} = \wp_{\triangleright}+3, \\ (\eth_{\triangleright}-\gimel_{\triangleright},\eth_{\triangleright}-\gimel_{\triangleright}+2); & \wp_{\triangleright}+4 \leq \gimel_{\triangleright} \leq \eth_{\triangleright}-1. \end{cases}$$

Inner cycles vertices are represented as $r(v_1|\{u_1,u_3\}) = (1,3)$ and also for $r(v_2|\{u_1,u_3\}) = (1,2)$ and for $r(v_3|\{u_1,u_3\}) = (2,1)$. In general, we'll talk about two cases.

Case 1: For even $\eth_{\triangleright}$ i.e., $\eth_{\triangleright} = 2\wp_{\triangleright}$ with $\wp_{\triangleright} = 3$, then,

$$r(v_{\gimel_{\triangleright}}|\{u_1,u_3\}) = \begin{cases} (\gimel_{\triangleright}-1,\gimel_{\triangleright}-3); & 4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright}+1, \\ (\wp_{\triangleright}-1,\wp_{\triangleright}-1); & \gimel_{\triangleright} = \wp_{\triangleright}+2, \\ (\wp_{\triangleright}-2,\wp_{\triangleright}); & \wp_{\triangleright}+3 \leq \gimel_{\triangleright} \leq \eth_{\triangleright}. \end{cases}$$

Case 2: For even $\eth_{\triangleright}$ i.e., $\eth_{\triangleright} = 2\wp_{\triangleright}$ with $\wp_{\triangleright} \geq 4$, then,

$$r(v_{\gimel_{\triangleright}}|\{u_1,u_3\}) = \begin{cases} (\gimel_{\triangleright}-1,\gimel_{\triangleright}-3); & 4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright}+1, \\ (\wp_{\triangleright}-1,\wp_{\triangleright}-1); & \gimel_{\triangleright} = \wp_{\triangleright}+2, \\ (\wp_{\triangleright}-2,\wp_{\triangleright}); & \gimel_{\triangleright} = \wp_{\triangleright}+3, \\ (\eth_{\triangleright}-\gimel_{\triangleright}+1,\eth_{\triangleright}-\gimel_{\triangleright}+3); & \wp_{\triangleright}+4 \leq \gimel_{\triangleright} \leq \eth_{\triangleright}. \end{cases}$$

Case 3: For odd $\eth_{\triangleright}$ i.e., $\eth_{\triangleright} = 2\wp_{\triangleright}+1$ with $\wp_{\triangleright} \geq 3$, then,

$$r(v_{\gimel_{\triangleright}}|\{u_1,u_3\}) = \begin{cases} (\gimel_{\triangleright}-1,\gimel_{\triangleright}-3); & 4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright}+1, \\ (\gimel_{\triangleright}-2,\gimel_{\triangleright}-3); & \gimel_{\triangleright} = \wp_{\triangleright}+2, \\ (\wp_{\triangleright}-1,\wp_{\triangleright}); & \gimel_{\triangleright} = \wp_{\triangleright}+3, \\ (\eth_{\triangleright}-\gimel_{\triangleright}+1,\eth_{\triangleright}-\gimel_{\triangleright}+3); & \wp_{\triangleright}+4 \leq \gimel_{\triangleright} \leq \eth_{\triangleright}. \end{cases}$$

We observe that, $u_{\gimel_{\triangleright}} = v_{\gimel_{\triangleright}}$ with $4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright}$, $\wp_{\triangleright} \geq 4$ have the same representation. v_{p+1}, u_p have the same representation where $\wp_{\triangleright}+3 \leq p \leq \eth_{\triangleright}-1$, $\eth_{\triangleright}$ is even and $\eth_{\triangleright} \geq 8$, $\wp_{\triangleright} \geq 4$. v_1 and $u_{\eth_{\triangleright}-1}$ have the same representation for $\wp_{\triangleright} \geq 3$ but, $\{u_1,u_3\}$ resolve the inner cycle vertices and $\{u_1,u_3\}$ also resolve the outer cycle vertices. We take $u_{\wp_{\triangleright}+1}$ with $\wp_{\triangleright} \geq 3$ to resolve the outer cycle vertices with the same representation.

$$d(u_{\wp_{\triangleright}+1},u_p) = \begin{cases} \wp_{\triangleright}-p; & p = 1 \ (\eth_{\triangleright} \text{ is even}), \\ \wp_{\triangleright}; & p = 1 \ (\eth_{\triangleright} \text{ is odd}), \\ \wp_{\triangleright}-p+1; & 2 \leq p \leq \wp_{\triangleright}, \\ p-\wp_{\triangleright}-1; & \wp_{\triangleright}+2 \leq p \leq \eth_{\triangleright}-1. \end{cases}$$

$d(u_{\wp_{\triangleright}+1},v_1) = \eth_{\triangleright}-\wp_{\triangleright}$ and the inner cycle distances are,

$$d(u_{\wp_{\triangleright}+1},v_p) = \begin{cases} \wp_{\triangleright}-p+2; & 2 \leq p \leq \wp_{\triangleright}, \\ 1; & p = \wp_{\triangleright}+1, \\ p-\wp_{\triangleright}-1; & \wp_{\triangleright}+2 \leq p \leq \eth_{\triangleright}. \end{cases}$$

We observe that, $u_{\wp_{\triangleright}+1}$ distinguish the vertices in $V(A_{\eth_{\triangleright}}.e)$, which were distinguished neither by u_1 nor by u_3. Hence, $\Re = \{u_1,u_3,u_{\wp_{\triangleright}+1}\}$ distinguish all vertices in $A_{\eth_{\triangleright}}.e$. This suggests that $dim(A_{\eth_{\triangleright}}.e) \leq 3$.

Conversely, we show that $dim(A_{\eth_{\triangleright}}.e) \geq 3$. On the contrary, assume that $dim(A_{\eth_{\triangleright}}.e) = 2$; we discuss few cases:

Case 1: We can fix u_1 as a basis vertex once both vertices are in the outer cycle. If the second vertex is $u_{\gimel_\triangleright}$, then,

(a) when $2 \leq \gimel_\triangleright \leq \wp_\triangleright - 1$, then we observe that,

$$r(u_{\gimel_\triangleright+1}|\{u_1, u_{\gimel_\triangleright}\}) = (\gimel_\triangleright, 1) = r(v_{\gimel_\triangleright+1}|\{u_1, u_{\gimel_\triangleright}\}.$$

(b) For even $\eth_\triangleright$ and $\gimel_\triangleright = \wp_\triangleright + 1$, $r(v_{\wp_\triangleright+3}|\{u_1, u_{\gimel_\triangleright}\}) = r(u_{\wp_\triangleright-1}|\{u_1, u_{\gimel_\triangleright}\})$ and $r(v_{\wp_\triangleright+2}|\{u_1, u_{\gimel_\triangleright}\}) = r(u_{\wp_\triangleright}|\{u_1, u_{\gimel_\triangleright}\})$.
For odd $\eth_\triangleright$ and $\gimel_\triangleright = \wp_\triangleright + 1$, $r(u_{\wp_\triangleright}|\{u_1, u_{\gimel_\triangleright}\}) = r(u_{\wp_\triangleright+2}|\{u_1, u_{\gimel_\triangleright}\})$ and $r(u_{\wp_\triangleright-1}|\{u_1, u_{\gimel_\triangleright}\}) = r(u_{\wp_\triangleright+3}|\{u_1, u_{\gimel_\triangleright}\})$ and $r(v_{\wp_\triangleright-1}|\{u_1, u_{\gimel_\triangleright}\}) = r(v_{\wp_\triangleright+4}|\{u_1, u_{\gimel_\triangleright}\})$ and $r(v_{\wp_\triangleright}|\{u_1, u_{\gimel_\triangleright}\}) = r(v_{\wp_\triangleright+3}|\{u_1, u_{\gimel_\triangleright}\})$ and $r(v_{\wp_\triangleright+1}|\{u_1, u_{\gimel_\triangleright}\}) = r(v_{\wp_\triangleright+2}|\{u_1, u_{\gimel_\triangleright}\})$.

Case 2: If we fix u_1 and $v_{\gimel_\triangleright}$ as the second vertex when one of these present in the outer cycle vertex and next is again in the inner cycle then,

(a) if $\gimel_\triangleright = 1$, then $r(v_3|\{u_1, v_{\gimel_\triangleright}\}) = r(v_{\eth_\triangleright-1}|\{u_1, v_{\gimel_\triangleright}\})$ and $r(v_4|\{u_1, v_{\gimel_\triangleright}\}) = r(v_{\eth_\triangleright-2}|\{u_1, v_{\gimel_\triangleright}\})$ and $r(u_2|\{u_1, v_{\gimel_\triangleright}\}) = r(u_{\eth_\triangleright-1}|\{u_1, v_{\gimel_\triangleright}\})$ and $r(u_3|\{u_1, v_{\gimel_\triangleright}\}) = r(u_{\eth_\triangleright-2}|\{u_1, v_{\gimel_\triangleright}\})$.

(b) If $\gimel_\triangleright = 2$, then $r(v_1|\{u_1, v_{\gimel_\triangleright}\}) = r(u_2|\{u_1, v_{\gimel_\triangleright}\})$.

(c) If $3 \leq \gimel_\triangleright \leq \wp_\triangleright + 1$, then $r(v_{\gimel_\triangleright-1}|\{u_1, v_{\gimel_\triangleright}\}) = r(u_{\gimel_\triangleright-1}|\{u_1, v_{\gimel_\triangleright}\})$.

Case 3: We can select v_1 as the basis vertex when both vertices belong to the inner cycle. If the vertex in second basis $v_{\gimel_\triangleright}$, then if $\gimel_\triangleright = \wp_\triangleright + 1$, $\eth_\triangleright$ is even, then $r(v_2|\{v_1, v_{\gimel_\triangleright}\}) = r(v_{\eth_\triangleright}|\{v_1, v_{\gimel_\triangleright}\})$. If $\eth_\triangleright$ is odd, $\gimel_\triangleright = \wp_\triangleright + 1$, then $r(u_1|\{v_1, v_{\gimel_\triangleright}\}) = r(v_{\eth_\triangleright}|\{v_1, v_{\gimel_\triangleright}\})$.

All of the cases discussed above indicate that two vertices are insufficient to resolve a problem through contraction $dim(A_{\eth_\triangleright}.e) \geq 3$, which yield $dim(A_{\eth_\triangleright}.e) = 3$. We know that $dim(A_{\eth_\triangleright}) = 3$, so this shows that $dim(A_{\eth_\triangleright}.e) = dim(A_{\eth_\triangleright})$. □

In Figure 1, (a) Antiprism graph $A_{\eth_\triangleright}$ and (b) Edge contracted Antiprism graph $A_{\eth_\triangleright}.e$ are discussed.

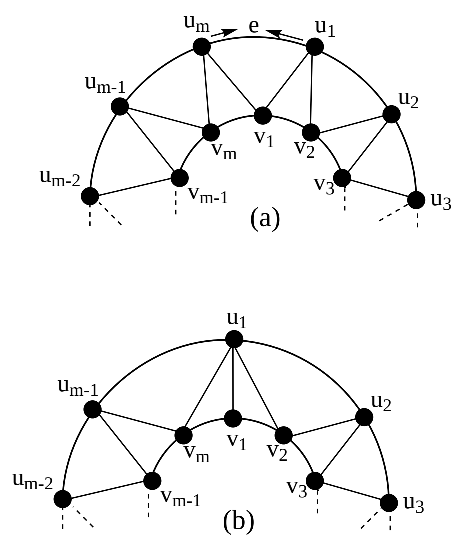

Figure 1: (a) Antiprism graph $A_{\eth_\triangleright}$. (b) Edge contracted Antiprism graph $A_{\eth_\triangleright}.e$.

Theorem 2.2. *Let $A_{\eth_\triangleright}$ with $\eth_\triangleright \geqslant 3$ be an Antiprism graph and $A_{\eth_\triangleright}.e$ be the inner edge contracted Antiprism graph, then; $dim(A_{\eth_\triangleright}.e) = dim(A_{\eth_\triangleright})$ for $\eth_\triangleright \geqslant 3$.*

Theorem 2.3. *Let $A_{\eth_\triangleright}$ with $\eth_\triangleright \geqslant 3$ be an Antiprism graph and $A_{\eth_\triangleright}.e$ be the middle edge contracted Antiprism graph, then; $dim(A_{\eth_\triangleright}.e) = dim(A_{\eth_\triangleright})$ for $\eth_\triangleright \geqslant 3$.*

Proof. Suppose $\eth_\triangleright = 2\wp_\triangleright$ or $\eth_\triangleright = 2\wp_\triangleright + 1$ for $\eth_\triangleright$, even and odd where $\wp_\triangleright \in \mathbb{Z}^+$. Resolving set for $V(A_{\eth_\triangleright}.e)$ is $\{u_1, u_2, u_3\}$ for $\eth_\triangleright = 3, 4, 5$. For $\eth_\triangleright \geqslant 6$, we will prove that RS is $\{u_1, u_3, u_{\wp_\triangleright+1}\}$. Firstly, we give vertices representation in $A_{\eth_\triangleright}.e$ with respect to $\{u_1, u_3\}$. We can observe that $r(u_2|\{u_1, u_3\}) = (1, 1)$ and $r(u_1|\{u_1, u_3\}) = (0, 2)$ and for $r(u_3|\{u_1, u_3\}) = (2, 0)$, and generally speaking, the outer cycle vertices are represented as follows:

Case 1: For even $\eth_\triangleright$ .i.e, $\eth_\triangleright = 2\wp_\triangleright$ with $\wp_\triangleright = 3$, then,

$$r(u_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \gimel_\triangleright - 3); & \wp_\triangleright + 2 \leq \gimel_\triangleright \leq \eth_\triangleright. \end{cases}$$

Case 2: For even $\eth_\triangleright$ i.e, $\eth_\triangleright = 2\wp_\triangleright$ with $\wp_\triangleright \geq 4$, then,

$$r(u_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \gimel_\triangleright - 3); & \wp_\triangleright + 2 \leq \gimel_\triangleright \leq \wp_\triangleright + 3, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \eth_\triangleright - \gimel_\triangleright + 3); & \wp_\triangleright + 4 \leq \gimel_\triangleright \leq \eth_\triangleright. \end{cases}$$

Case 3: For odd $\eth_\triangleright$ i.e, $\eth_\triangleright = 2\wp_\triangleright + 1$ with $\wp_\triangleright \geq 3$, then,

$$r(u_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \gimel_\triangleright - 3); & \wp_\triangleright + 2 \leq \gimel_\triangleright \leq \wp_\triangleright + 3, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \eth_\triangleright - \gimel_\triangleright + 3); & \wp_\triangleright + 4 \leq \gimel_\triangleright \leq \eth_\triangleright. \end{cases}$$

The inner cycle vertices representation are, $r(v_1|\{u_1, u_3\}) = (1, 3)$, $r(v_2|\{u_1, u_3\}) = (1, 1)$, $r(v_3|\{u_1, u_3\}) = (2, 1)$, and few cases to be discussed here.

Case 1: For $\eth_\triangleright$ even i.e. $\eth_\triangleright = 2\wp_\triangleright$ with $\wp_\triangleright = 3$, then,

$$r(v_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\wp_\triangleright - 1, \wp_\triangleright - 1); & \wp_\triangleright + 2 \leq \gimel_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 2: For $\eth_\triangleright$ even i.e. $\eth_\triangleright = 2\wp_\triangleright$ with $\wp_\triangleright = 4$, then,

$$r(v_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\wp_\triangleright - 1, \wp_\triangleright - 1); & \gimel_\triangleright = \wp_\triangleright + 2, \\ (\wp_\triangleright - 2, \wp_\triangleright); & \wp_\triangleright + 3 \leq \gimel_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 3: For $\eth_\triangleright$ even i.e. $\eth_\triangleright = 2\wp_\triangleright$ with $\wp_\triangleright \geq 5$, then,

$$r(v_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\wp_\triangleright - 1, \wp_\triangleright - 1); & \gimel_\triangleright = \wp_\triangleright + 2, \\ (\wp_\triangleright - 2, \wp_\triangleright); & \gimel_\triangleright = \wp_\triangleright + 3, \\ (\eth_\triangleright - \gimel_\triangleright + 1, \eth_\triangleright - \gimel_\triangleright + 3); & \wp_\triangleright + 4 \leq \gimel_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 4: For odd $\eth_\triangleright$ i.e. $\eth_\triangleright = 2\wp_\triangleright + 1$ with $\wp_\triangleright = 3$, then,

$$r(v_{\gimel_\triangleright}|\{u_1, u_3\}) = \begin{cases} (\gimel_\triangleright - 1, \gimel_\triangleright - 3); & 4 \leq \gimel_\triangleright \leq \wp_\triangleright + 1, \\ (\gimel_\triangleright - 2, \gimel_\triangleright - 3); & \gimel_\triangleright = \wp_\triangleright + 2, \\ (\wp_\triangleright - 1, \wp_\triangleright); & \wp_\triangleright + 3 \leq \gimel_\triangleright \leq \eth_\triangleright - 1. \end{cases}$$

Case 5: For odd $\eth_{\triangleright}$ i.e. $\eth_{\triangleright} = 2\wp_{\triangleright} + 1$ with $\wp_{\triangleright} \geq 4$, then,

$$r(v_{\gimel_{\triangleright}}|\{u_1, u_3\}) = \begin{cases} (\gimel_{\triangleright} - 1, \gimel_{\triangleright} - 3); & 4 \leq \gimel_{\triangleright} \leq \wp_{\triangleright} + 1, \\ (\gimel_{\triangleright} - 2, \gimel_{\triangleright} - 3); & \gimel_{\triangleright} = \wp_{\triangleright} + 2, \\ (\wp_{\triangleright} - 1, \wp_{\triangleright}); & \gimel_{\triangleright} = \wp_{\triangleright} + 3, \\ (\eth_{\triangleright} - \gimel_{\triangleright} + 1, \eth_{\triangleright} - \gimel_{\triangleright} + 3); & \wp_{\triangleright} + 4 \leq \gimel_{\triangleright} \leq \eth_{\triangleright} - 1. \end{cases}$$

We observed that, here $u_{\gimel_{\triangleright}} = v_{\gimel_{\triangleright}}$ with $\wp_{\triangleright} + 3 \leq \gimel_{\triangleright} \leq \eth_{\triangleright} - 1$ where $\wp_{\triangleright} \geq 4$ and $v_1 = u_{\eth_{\triangleright}}$ have the same representation, but $\{u_1, u_3\}$ resolve inner cycle vertices and $\{u_1, u_3\}$ also resolve outer cycle vertices. We take $u_{\wp_{\triangleright}+1}$ to resolve the outer cycle vertices which have the same representation.

$$d(u_{\wp_{\triangleright}+1}, u_p) = \begin{cases} \wp_{\triangleright} - p + 1; & 1 \leq p \leq \wp_{\triangleright}, \\ p - \wp_{\triangleright} - 1; & \wp_{\triangleright} + 2 \leq p \leq \eth_{\triangleright}. \end{cases}$$

$d(u_{\wp_{\triangleright}+1}, v_1) = \eth_{\triangleright} - \wp_{\triangleright}$ and the inner cycle vertices distances are,

$$d(u_{\wp_{\triangleright}+1}, v_p) = \begin{cases} \wp_{\triangleright} - p + 1; & 2 \leq p \leq \wp_{\triangleright}, \\ 1; & p = \wp_{\triangleright} + 1, \\ p - \wp_{\triangleright}; & \wp_{\triangleright} + 2 \leq p \leq \eth_{\triangleright} - 1. \end{cases}$$

We see that, $u_{\wp_{\triangleright}+1}$ distinguish the vertices in $V(A_{\eth_{\triangleright}}.e)$, which were distinguished neither by u_1 nor by u_3. Hence, $\Re = \{u_1, u_3, u_{\wp_{\triangleright}+1}\}$ resolves all vertices in $A_{\eth_{\triangleright}}.e$. This implies that $dim(A_{\eth_{\triangleright}}.e) \leq 3$. Conversely, we will show that $dim(A_{\eth_{\triangleright}}.e) \geq 3$. On the contrary, suppose that $dim(A_{\eth_{\triangleright}}.e) = 2$, then three cases will be discussed:

Case 1: When both the vertices belong to the outer cycle, then we can take u_1. Now, if the second element is $u_{\gimel_{\triangleright}}$, then,

(a) For even $\eth_{\triangleright}$ and $\gimel_{\triangleright} = \wp_{\triangleright} + 1$, $r(v_{\wp_{\triangleright}-1}|\{u_1, u_{\gimel_{\triangleright}}\}) = r(u_{\wp_{\triangleright}-1}|\{u_1, u_{\gimel_{\triangleright}}\}$,

(b) For odd $\eth_{\triangleright}$ and $\gimel_{\triangleright} = \wp_{\triangleright} + 1$, $r(v_{\wp_{\triangleright}+1}|\{u_1, u_{\gimel_{\triangleright}}\}) = r(u_{\wp_{\triangleright}+2}|\{u_1, u_{\gimel_{\triangleright}}\})$.

Case 2: We fix u_1 as the basis vertex and $v_{\gimel_{\triangleright}}$ as the second basis vertex when outer cycle have one vertex and second is in the inner cycle again, then,

(a) If $\gimel_{\triangleright} = 2$ then, $r(v_1|\{u_1, v_{\gimel_{\triangleright}}\}) = r(u_{\eth_{\triangleright}}|\{u_1, v_{\gimel_{\triangleright}}\})$.

Case 3: We can select v_1 as the basis vertex when both vertices are in the inner cycle. If second basis vertex is $v_{\gimel_{\triangleright}}$, then,

(a) If $\eth_{\triangleright}$ is odd and $\gimel_{\triangleright} = \wp_{\triangleright} + 1$, then $r(v_2|\{v_1, v_{\gimel_{\triangleright}}\}) = r(u_{\eth_{\triangleright}-1}|\{v_1, v_{\gimel_{\triangleright}}\})$.

(b) If $\eth_{\triangleright}$ is even and $\gimel_{\triangleright} = \wp_{\triangleright} + 1$, then $r(v_{\wp_{\triangleright}}|\{v_1, v_{\gimel_{\triangleright}}\}) = r(u_{\wp_{\triangleright}+1}|\{v_1, v_{\gimel_{\triangleright}}\})$.

All of the cases discussed above indicate that two vertices are insufficient to resolve a problem through contraction $dim(A_{\eth_{\triangleright}}.e) \geq 3$, which yield, $dim(A_{\eth_{\triangleright}}.e) = 3$. We know that $dim(A_{\eth_{\triangleright}}) = 3$, so this shows that $dim(A_{\eth_{\triangleright}}.e) = dim(A_{\eth_{\triangleright}})$. □

In Figure 2, (a) Antiprism graph $A_{\eth_{\triangleright}}$ and (b) Edge contracted Antiprism graph $A_{\eth_{\triangleright}}.e$ are discussed.

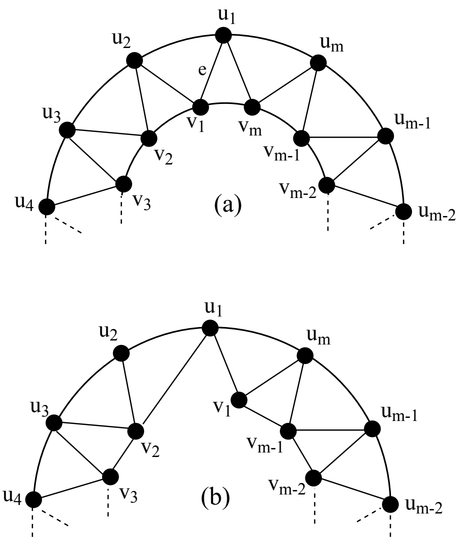

Figure 2: (a) Antiprism graph $A_{\eth_{\triangleright}}$. (b) Edge contracted Antiprism graph $A_{\eth_{\triangleright}}.e$.

3 Metric Dimension of Edge Contracted Petersen Graph

The generalized Petersen graph $\mathrm{P}(\eth_{\triangleright}, \wp_{\triangleright})$, $\eth_{\triangleright} \geq 3$ contain vertices $V = \{y_1, y_2, \dots y_n, x_1, x_2, \dots x_n\}$ and an edges $E = \{y_{\gimel_{\triangleright}} y_{\gimel_{\triangleright}+1}, y_{\gimel_{\triangleright}} x_{\gimel_{\triangleright}}, x_{\gimel_{\triangleright}} x_{\gimel_{\triangleright}+\wp_{\triangleright}}$ | with indices taken modulo $\eth_{\triangleright}\}$. To achieve our target, we name the cycle produced by $\{y_1, y_2, \dots y_{\eth_{\triangleright}}\}$ outer cycle and $\{x_1, x_2, \dots x_{\eth_{\triangleright}}\}$ inner cycle. It should be noted that the core issue is the selection of suitable basis vertices.

Theorem 3.1. *Let* $\mathrm{P}(\eth_{\triangleright}, 2)$ *be the generalized Petersen graph and* $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ *be the outer EC of generalized Petersen graph then;* $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = dim(\mathrm{P}(\eth_{\triangleright}, 2))$ *for* $\eth_{\triangleright} \geq 5$.

Proof. We will to show that, three vertices appropriately chosen suffices to distinguish all vertices in $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$. We discuss few cases here:

Case 1: $\eth_{\triangleright} \equiv 0(\mod 4)$
We take $\eth_{\triangleright} = 4\mathrm{u}_{\triangleright}$, $\mathrm{u}_{\triangleright} \geq 2$ and $\mathrm{u}_{\triangleright} \in \mathbb{Z}^{+}$. Here, $\Re_1 = \{x_1, x_2, x_3\}$ resolves $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$. Indeed, x_1 and x_2 resolve inner and outer cycle vertices. To show that $\Re_1 = \{x_1, x_2, x_3\}$ resolves vertices $(\mathrm{P}(\eth_{\triangleright}, 2).e)$, first we give $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ vertices representation regards as $\Re = \{x_1, x_2\}$. Outer cycle vertices representation are $r(y_1|\Re) = (1, 2)$ and also $r(y_2|\Re) = (2, 1)$,

$$r(y_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 2); & \mathrm{u}_{\triangleright} + 1 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \end{cases}$$

and

$$r(y_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}+1); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 2); & \mathrm{u}_{\triangleright} + 1 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 2, \\ (2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \gimel_{\triangleright} + 1); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright} - 1, \end{cases}$$

and in the inner cycle,

$$r(x_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+2, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+1); & \mathrm{u}_{\triangleright}+1 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}, \end{cases}$$

and

$$r(x_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}+2); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3); & \mathrm{u}_{\triangleright}+1 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}-2, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}-1. \end{cases}$$

In the inner cycles, take note that any two vertices have not matching representation. Additionally, neither inner nor outer cycles vertices share a representation, but outer cycle have, $r(y_{3+\gimel_{\triangleright}}|\Re) = r(y_{\eth_{\triangleright}-\gimel_{\triangleright}}|\Re)$ for $\gimel_{\triangleright} = 2, 4, \ldots 2\mathrm{u}_{\triangleright}-2$ and $(y_3|\Re) = r(y_{\eth_{\triangleright}-1}|\Re)$.

Vertex x_3 resolve the same representation as,

$$d(x_3, y_{3+\gimel_{\triangleright}}) = \left\lfloor \frac{3+\gimel_{\triangleright}}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-\gimel_{\triangleright}}) = \left\lfloor \frac{3+\gimel_{\triangleright}}{2} \right\rfloor + 1, \text{ for } \gimel_{\triangleright} = 2, 4, \ldots 2\mathrm{u}_{\triangleright}-4,$$

and $d(x_3, y_3) = \left\lfloor \frac{3}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-1}) = \left\lfloor \frac{3}{2} \right\rfloor + 2$ and $d(x_3, y_{2\mathrm{u}_{\triangleright}+2}) = d(x_3, y_{2\mathrm{u}_{\triangleright}+1}) + 1$. This suggest that $\Re_1 = \{x_1, x_2, x_3\}$ resolve vertices of $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ which means $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$ when $\eth_{\triangleright} \equiv 0(\mod 4)$.

Conversely, we will express $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$. Assume contrary that $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 2$, then there are three cases to be discussed.

Case i: We can fix y_1 as a basis vertex once both vertices are in the outer cycle. At least two vertices will have the same representation which is a contraction, if the second element in the basis is $y_{\gimel_{\triangleright}}$.

(a) If $\gimel_{\triangleright} = 2$, then $r(y_{\eth_{\triangleright}-1}|\{y_1, y_{\gimel_{\triangleright}}\}) = r(x_1|\{y_1, y_{\gimel_{\triangleright}}\}) = (1, 2)$.

(b) If $\gimel_{\triangleright} = 3, 5, \ldots 2\mathrm{u}_{\triangleright}-1$, then $r(x_{\gimel_{\triangleright}+1}|\{y_1, y_{\gimel_{\triangleright}}\}) = r(x_{\gimel_{\triangleright}+2}|\{y_1, y_{\gimel_{\triangleright}}\})$. Further if, $\gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}+1$, then $r(x_{2\mathrm{u}_{\triangleright}}|\{y_1, y_{\gimel_{\triangleright}}\}) = r(x_{2\mathrm{u}_{\triangleright}+2}|\{y_1, y_{\gimel_{\triangleright}}\})$.

(c) If $\gimel_{\triangleright} = 4, 6, \ldots 2\mathrm{u}_{\triangleright}$, then $r(x_2|\{y_1, y_{\gimel_{\triangleright}}\}) = \left(2, \frac{\gimel_{\triangleright}}{2}\right) = r(x_3|\{y_1, y_{\gimel_{\triangleright}}\})$.

Case ii: Once both vertices belong to the outer cycle, we can take y_1. If the second vertex is $y_{\gimel_{\triangleright}}$, at least two vertices will have the same representation, which is a contraction.

(a) If $\gimel_{\triangleright} = 1$, then $r(y_2|\{y_1, x_{\gimel_{\triangleright}}\}) = r(y_{\eth_{\triangleright}-1}|\{y_1, x_{\gimel_{\triangleright}}\})$.

(b) If $\gimel_{\triangleright} = 2$, then $r(x_3|\{y_1, x_{\gimel_{\triangleright}}\}) = r(x_{\eth_{\triangleright}-1}|\{y_1, x_{\gimel_{\triangleright}}\})$.

(c) If $\gimel_{\triangleright} = 3$, then $r(x_2|\{y_1, x_{\gimel_{\triangleright}}\}) = (2, 3)$. For $\gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}+1$, $r(x_2|\{y_1, x_{\gimel_{\triangleright}}\}) = r(x_{\eth_{\triangleright}}|\{y_1, x_{\gimel_{\triangleright}}\})$.

(d) If $\gimel_{\triangleright} = 4, 6, \ldots, 2\mathrm{u}_{\triangleright}$, then $r(y_3|\{y_1, x_{\gimel_{\triangleright}}\}) = r(x_{\eth_{\triangleright}}|\{y_1, x_{\gimel_{\triangleright}}\})$.

Case iii: We can take x_1, when both vertices belong the inner cycle. If $x_{\gimel_{\triangleright}}$ is the second vertex, then at least two elements with the same representation will be found, resulting in a contraction.

(a) If $\gimel_{\triangleright} = 2$, then $r(y_3|\{x_1, x_{\gimel_{\triangleright}}\}) = r(y_{\eth_{\triangleright}-1}|\{x_1, x_{\gimel_{\triangleright}}\})$.

(b) If $\gimel_{\triangleright} = 3, 5, \ldots, 2\mathrm{u}_{\triangleright}-1$, then $r(y_{\gimel_{\triangleright}+1}|\{x_1, x_{\gimel_{\triangleright}}\}) = r(y_{\gimel_{\triangleright}+2}|\{x_1, x_{\gimel_{\triangleright}}\})$ and for $\gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}+1$, $r(x_{2\mathrm{u}_{\triangleright}}|\{x_1, x_{\gimel_{\triangleright}}\}) = r(x_{2\mathrm{u}_{\triangleright}+2}|\{x_1, x_{\gimel_{\triangleright}}\})$.

(c) If $\gimel_{\triangleright} = 4, 6, \ldots, 2\mathrm{u}_{\triangleright}$, then $r(y_{\gimel_{\triangleright}-2}|\{x_1, x_{\gimel_{\triangleright}}\}) = r(y_{\gimel_{\triangleright}-1}|\{x_1, x_{\gimel_{\triangleright}}\})$. All above cases suggest that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$. Which means that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$, when $\eth_{\triangleright} \equiv 0(\mod 4)$.

Case 2: $\eth_\triangleright \equiv 2 \pmod 4$

We see that $\eth_\triangleright = 4\mathrm{u}_\triangleright + 2$ with $\mathrm{u}_\triangleright \geq 1$ and $\mathrm{u}_\triangleright \in \mathbb{Z}^+$. Again in this, we will show that $\{x_1, x_2, x_3\}$ resolve $V(\mathrm{P}(\eth_\triangleright, 2).e)$. Again x_1 and x_2 resolve the inner cycle vertices and they will also distinguish outer cycle vertices and the inner cycle. To show that $\{x_1, x_2, x_3\}$ resolve vertices of $V(\mathrm{P}(\eth_\triangleright, 2).e)$. We give $V(\mathrm{P}(\eth_\triangleright, 2).e)$ vertices representation as regards $\{x_1, x_2\}$. Outer cycle vertices representation are, $(y_1|\{x_1, x_2\} = (1, 2)$ and $(y_2|\{x_1, x_2\} = (2, 1)$,

$$r(y_{2\gimel_\triangleright}|\Re) = \begin{cases} (\gimel_\triangleright + 1, \gimel_\triangleright); & 2 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright + 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 3, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \mathrm{u}_\triangleright + 3 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright, \end{cases}$$

and

$$r(y_{2\gimel_\triangleright+1}|\Re) = \begin{cases} (\gimel_\triangleright + 1, \gimel_\triangleright + 1); & 1 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 2, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \mathrm{u}_\triangleright + 1 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright - 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 2, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 2); & \gimel_\triangleright = 2\mathrm{u}_\triangleright. \end{cases}$$

And inner cycle have,

$$r(x_{2\gimel_\triangleright}|\Re) = \begin{cases} (\gimel_\triangleright + 2, \gimel_\triangleright - 1); & 2 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright + 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 4, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 2); & \mathrm{u}_\triangleright + 2 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright + 1, \end{cases}$$

and

$$r(x_{2\gimel_\triangleright+1}|\Re) = \begin{cases} (\gimel_\triangleright, \gimel_\triangleright + 2); & 1 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 1, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 4); & \mathrm{u}_\triangleright + 1 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright - 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 1, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \gimel_\triangleright = 2\mathrm{u}_\triangleright. \end{cases}$$

Once more, in the inner cycle vertices, no two have the same representation in this instance. Additionally, same represented vertices are not present in the inner and outer cycle. But the outer cycle have, $r(y_{3+\gimel_\triangleright}|\Re) = r(y_{\eth_\triangleright - \gimel_\triangleright}|\Re)$ for $\gimel_\triangleright = 2, 4, \ldots 2\mathrm{u}_\triangleright - 2$ and $r(y_3|\Re) = r(y_{\eth_\triangleright - 1}|\Re)$ Vertex x_3 resolve same representation vertices as,

$$d(x_3, y_3) = \left\lfloor \frac{3}{2} \right\rfloor \neq d(x_3, y_{\eth_\triangleright - 1}) = \left\lfloor \frac{3}{2} \right\rfloor + 2,$$

and

$$d(x_3, y_{3+\gimel_\triangleright}) = \left\lfloor \frac{3 + \gimel_\triangleright}{2} \right\rfloor \neq d(x_3, y_{\eth_\triangleright - \gimel_\triangleright}) = \left\lfloor \frac{3 + \gimel_\triangleright}{2} \right\rfloor + 2,$$

for $\gimel_\triangleright = 2, 4, \ldots, 2\mathrm{u}_\triangleright - 2$. This suggest that $\{x_1, x_2, x_3\}$ resolve vertices of $\mathrm{P}(\eth_\triangleright, 2).e)$ which means $dim(\mathrm{P}(\eth_\triangleright, 2).e) \leq 3$ when $\eth_\triangleright \equiv 2 \pmod 4$. On the other hand, from Case i, $dim(\mathrm{P}(\eth_\triangleright, 2).e) \leq 3$. Hence, $dim(\mathrm{P}(\eth_\triangleright, 2).e) = 3$ for $\eth_\triangleright \equiv 2 \pmod 4$.

Case 3: $\eth_\triangleright \equiv 1 \pmod 4$

We see that $\eth_\triangleright = 4\mathrm{u}_\triangleright + 1$, with $\mathrm{u}_\triangleright \geq 1$ and $\mathrm{u}_\triangleright \in \mathbb{Z}^+$. We can see that, RS is $\{x_1, x_2, y_3\}$ for standard Petersen graph $(\mathrm{P}(5, 2).e)$. $W = \{x_1, x_2, y_4\}$ resolve all vertices in $(\mathrm{P}(9, 2).e)$ as the represented vertices are, $r(y_1|W) = (1, 2, 3)$ and $r(y_2|W) = (2, 1, 2)$ and $r(y_3|W) = (2, 2, 1)$ and $r(y_4|W) = (3, 2, 0)$ and for $r(y_5|W) = (3, 3, 1)$ and $r(y_6|W) = (3, 3, 2)$ and $r(y_7|W) = (3, 3, 3)$ and $r(y_8|W) = (2, 2, 4)$ and $r(x_1|W) = (0, 3, 3)$ and $r(x_2|W) = (3, 0, 2)$ and $r(x_3|W) = (1, 3, 2)$ and $r(x_4|W) = (3, 1, 1)$ and for $r(x_5|W) = (2, 3, 2)$ and $r(x_6|W) = (2, 2, 2)$ and $r(x_7|W) = (3, 2, 3)$ and also $r(x_8|W) = (1, 3, 3)$ that is $r(x_9|W) = (3, 1, 3)$.

For $\eth_{\triangleright} \geq 13$, we will show that $\{x_1, x_2, y_{2\mathrm{u}_{\triangleright}-1}\}$ resolve vertices of $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ where, $\eth_{\triangleright} \equiv 1(\mod 4)$. For this, first we give represented vertices as regards $\{x_1, x_2\}$. The outer cycle vertices are represented as, $r(y_1|\Re) = (1,2)$ and $r(y_2|\Re) = (2,1)$,

$$r(y_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+1); & \mathrm{u}_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}-1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}, \end{cases}$$

$$r(y_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}+1); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2); & \mathrm{u}_{\triangleright}+1 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}-1. \end{cases}$$

Now in the inner cycle,

$$r(x_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+2, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}-1, \\ (\gimel_{\triangleright}+1, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}, \\ (\gimel_{\triangleright}-1, \gimel_{\triangleright}-1) & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+1, \\ (\gimel_{\triangleright}-3, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+2, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+1, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+4); & \mathrm{u}_{\triangleright}+3 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}-1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+1, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}, \end{cases}$$

and

$$r(x_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}+2); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}-1, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}+1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+1); & \mathrm{u}_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}. \end{cases}$$

Note that, $\{x_1, x_2\}$ resolves all but the following vertices y_3 and $y_{\eth_{\triangleright}-1}$, $y_{2\mathrm{u}_{\triangleright}-1}$ and $y_{2\mathrm{u}_{\triangleright}+5}$ and $x_{2\mathrm{u}_{\triangleright}+2}$, $y_{2\mathrm{u}_{\triangleright}+1}$ and $y_{2\mathrm{u}_{\triangleright}+2}$ and $y_{2\mathrm{u}_{\triangleright}+3}$, $x_{2\mathrm{u}_{\triangleright}-1}$ and $x_{2\mathrm{u}_{\triangleright}+4}$, $x_{2\mathrm{u}_{\triangleright}}$ and $y_{2\mathrm{u}_{\triangleright}+5}$, $y_{2\mathrm{u}_{\triangleright}}$ and $x_{2\mathrm{u}_{\triangleright}+3}$, $x_{2\mathrm{u}_{\triangleright}+1}$ and $y_{2\mathrm{u}_{\triangleright}+4}$.

We see that outer cycle vertices that have the same representation are located at various distances from $y_{2\mathrm{u}_{\triangleright}-1}$ which are $d(y_{2\mathrm{u}_{\triangleright}-1}, y_{2\mathrm{u}_{\triangleright}+5}) = 5$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}+2}) = 3$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}-1}) = 1$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}+4}) = 4$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}}) = 2$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}+5}) = 4$, $d(y_{2\mathrm{u}_{\triangleright}-1}, y_{2\mathrm{u}_{\triangleright}}) = 1$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}+3}) = 3$, $d(y_{2\mathrm{u}_{\triangleright}-1}, x_{2\mathrm{u}_{\triangleright}+1}) = 2$, $d(y_{2\mathrm{u}_{\triangleright}-1}, y_{2\mathrm{u}_{\triangleright}+4}) = 5$.

The aforementioned suggests that, RS $\{x_1, x_2, y_{2\mathrm{u}_{\triangleright}-1}\}$ for $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$ when $\eth_{\triangleright} \equiv 1(\mod 4)$ and $\eth_{\triangleright} \geq 13$. Hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$ for $\eth_{\triangleright} \equiv 1(\mod 4)$. Arguments express that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$ are analogous with Case 1 hence $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$ even for $\eth_{\triangleright} \equiv 3(\mod 4)$.

Case 4: $\eth_{\triangleright} \equiv 3(\mod 4)$
We write $\eth_{\triangleright} = 4\mathrm{u}_{\triangleright}+3$ with $\mathrm{u}_{\triangleright} \geq 1$ and $\mathrm{u}_{\triangleright} \in \mathbb{Z}^+$. It is not tough to see that, $\{x_1, x_2, y_3\}$ is a RS for $V(\mathrm{P}(7,2).e)$. For $\eth_{\triangleright} \equiv 3(\mod 4)$ and also here $\eth_{\triangleright} \geq 11$, we next express that $\{x_1, x_2, y_{2\mathrm{u}_{\triangleright}+1}\}$ resolves $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$. We give representation of the vertices as regards $\{x_1, x_2\}$. Outer cycle vertices are represented as $r(y_1|\Re) = (1,2)$ and for $r(y_2|\Re) = (2,1)$,

$$r(y_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}+1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+4); & \mathrm{u}_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}+1, \end{cases}$$

and

$$r(y_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}+1); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}+1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+3); & \mathrm{u}_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}. \end{cases}$$

Now inner cycle have,

$$r(x_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+2, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+1 \\ (\gimel_{\triangleright}-2, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+2, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+5); & \mathrm{u}_{\triangleright}+3 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+4); & \gimel_{\triangleright} = 2\mathrm{u}_{\triangleright}+1, \end{cases}$$

$$r(x_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}+2); & 1 \leq \gimel_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}); & \gimel_{\triangleright} = \mathrm{u}_{\triangleright}+1, \\ (2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+4, 2\mathrm{u}_{\triangleright}-\gimel_{\triangleright}+2); & \mathrm{u}_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2\mathrm{u}_{\triangleright}+1. \end{cases}$$

Again in this case, $\{x_1, x_2\}$ resolve all the vertices in $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ but the following. y_3 and $y_{\eth_{\triangleright}-1}$, $y_{2\mathrm{u}_{\triangleright}}$ and $x_{2\mathrm{u}_{\triangleright}+2}$, $y_{2\mathrm{u}_{\triangleright}+1}$ and $y_{2\mathrm{u}_{\triangleright}+5}$ and $x_{2\mathrm{u}_{\triangleright}+3}$, $x_{2\mathrm{u}_{\triangleright}+4}$ and $y_{2\mathrm{u}_{\triangleright}+6}$. It can be seen that the outer cycle vertices with the same representation are resolve from $u_{2\mathrm{u}_{\triangleright}+1}$ at different distances. $d(y_{2\mathrm{u}_{\triangleright}+1}, y_{2\mathrm{u}_{\triangleright}}) = 1$, $d(y_{2\mathrm{u}_{\triangleright}+1}, x_{2\mathrm{u}_{\triangleright}+2}) = 2$, $d(y_{2\mathrm{u}_{\triangleright}+1}, y_{2\mathrm{u}_{\triangleright}+5}) = 4$, $d(y_{2\mathrm{u}_{\triangleright}+1}, x_{2\mathrm{u}_{\triangleright}+3}) = 2$, $d(y_{2\mathrm{u}_{\triangleright}+1}, x_{2\mathrm{u}_{\triangleright}+4}) = 3$, $d(y_{2\mathrm{u}_{\triangleright}+1}, y_{2\mathrm{u}_{\triangleright}+6}) = 5$.

Hence, RS for $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$ is $\{x_1, x_2, x_{2\mathrm{u}_{\triangleright}+1}\}$ when $\eth_{\triangleright} \equiv 3(\mod 4)$ and like the same above discussed cases, arguments similar to Case 1 suggest that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$ hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$ for $\eth_{\triangleright} \equiv 3(\mod 4)$. We notice that, to resolve all vertices in $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ only three vertices suffices, for any value of $\eth_{\triangleright} \geq 5$ which express that generalized Petersen graph $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ constitute constant MD of few families of graphs. Since $dim(\mathrm{P}(\eth_{\triangleright}, 2)) = 3$, this shows that $dim(\mathrm{P}(\eth_{\triangleright}, 2)) = dim(\mathrm{P}(\eth_{\triangleright}, 2).e)$.

In Figure 3, (a) Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2)$ and (b) Edge contracted Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2).e$ are discussed.

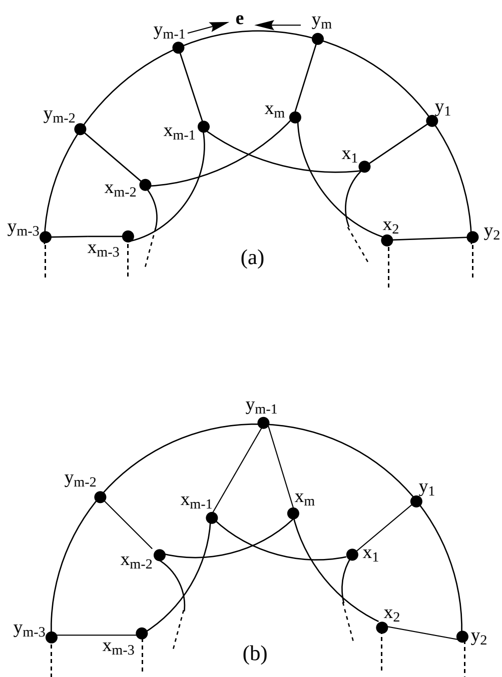

Figure 3: (a) Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2)$. (b) Edge contracted Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2).e$.

□

Theorem 3.2. *Let* $\mathrm{P}(\eth_{\triangleright}, 2)$ *be the generalized Petersen graph and* $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ *be the middle edge contracted generalized Petersen graph, then;* $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = dim(\mathrm{P}(\eth_{\triangleright}, 2))$ *for* $\eth_{\triangleright} \geq 5$.

Proof. We will show that to resolve all vertices in $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$, three vertices appropriately chosen. We shall discuss few cases.

Case 1: $\eth_{\triangleright} \equiv 0 \pmod 4$)

We see that $\eth_{\triangleright} = 4\mathrm{u}_{\triangleright}$, $\mathrm{u}_{\triangleright} \geq 2$ and $\mathrm{u}_{\triangleright} \in \mathbb{Z}^{+}$. Here, $\Re_1 = \{x_1, x_2, x_3\}$ resolves $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$. Indeed, x_1 and x_2 resolve outer cycle vertices and the inner cycles. To show that $\Re_1 = \{x_1, x_2, x_3\}$ resolves vertices of $(\mathrm{P}(\eth_{\triangleright}, 2).e)$, firstly representations of the vertices in $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ as regards $\Re = \{x_1, x_2\}$.

Outer cycle vertices representation are, $r(y_1|\Re) = (1, 2)$ and also $r(y_2|\Re) = (2, 1)$,

$$r(y_{2\beth_{\triangleright}}|\Re) = \begin{cases} (\beth_{\triangleright} + 1, \beth_{\triangleright}); & 2 \leq \beth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 2); & \mathrm{u}_{\triangleright} + 1 \leq \beth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 1); & \beth_{\triangleright} = 2\mathrm{u}_{\triangleright}, \end{cases}$$

and

$$r(y_{2\beth_{\triangleright}+1}|\Re) = \begin{cases} (\beth_{\triangleright} + 1, \beth_{\triangleright} + 1); & 1 \leq \beth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 2); & \mathrm{u}_{\triangleright} + 1 \leq \beth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 2, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 1); & \beth_{\triangleright} = 2\mathrm{u}_{\triangleright} - 1. \end{cases}$$

And, in the inner cycles,

$$r(x_{2\beth_{\triangleright}}|\Re) = \begin{cases} (\beth_{\triangleright} + 2, \beth_{\triangleright} - 1); & 2 \leq \beth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 3, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 1); & \mathrm{u}_{\triangleright} + 1 \leq \beth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \end{cases}$$

and

$$r(x_{2\beth_{\triangleright}+1}|\Re) = \begin{cases} (\beth_{\triangleright}, \beth_{\triangleright} + 2); & 1 \leq \beth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright}, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 3); & \mathrm{u}_{\triangleright} + 1 \leq \beth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 2, \\ (2\mathrm{u}_{\triangleright} - \beth_{\triangleright}, 2\mathrm{u}_{\triangleright} - \beth_{\triangleright} + 2); & \beth_{\triangleright} = 2\mathrm{u}_{\triangleright} - 1. \end{cases}$$

Note that inner cycle vertices have no same representation. Inner cycles have no vertices and outer cycle which have the common representation. But in the outer cycle, $r(y_{3+\beth_{\triangleright}}|\Re) = r(y_{\eth_{\triangleright}-\beth_{\triangleright}}|\Re)$ for $\beth_{\triangleright} = 2, 4, \ldots 2\mathrm{u}_{\triangleright} - 2$ and $(y_3|\Re) = r(y_{\eth_{\triangleright}-1}|\Re)$.

Vertex x_3 resolve the same representations of the vertices as,

$$d(x_3, y_{3+\beth_{\triangleright}}) = \left\lfloor \frac{3 + \beth_{\triangleright}}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-\beth_{\triangleright}}) = \left\lfloor \frac{3 + \beth_{\triangleright}}{2} \right\rfloor + 2, \text{ for } \beth_{\triangleright} = 2, 4, \ldots 2\mathrm{u}_{\triangleright} - 4,$$

and $d(x_3, y_3) = \left\lfloor \frac{3}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-1}) = \left\lfloor \frac{3}{2} \right\rfloor + 2$ and

$$d(x_3, y_{2\mathrm{u}_{\triangleright}+2}) = d(x_3, y_{2\mathrm{u}_{\triangleright}+1}) + 1.$$

This suggest that $\Re_1 = \{x_1, x_2, x_3\}$ resolves vertices of $(\mathrm{P}(\eth_{\triangleright}, 2).e)$, which means $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$ when $\eth_{\triangleright} \equiv 0 \pmod 4$).

Conversely, we will express, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$. Assume in contrary that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 2$, then there are few cases to be discussed.

Case i: If both vertices belong to outer cycle, then we take y_1. Now, if the second vertex is $y_{\gimel_\triangleright}$, then we will find at least two vertices with the same representation, a contraction.

(a) If $\gimel_\triangleright = 2$, then $r(y_{\eth_\triangleright}|\{y_1, y_{\gimel_\triangleright}\}) = r(x_1|\{y_1, y_{\gimel_\triangleright}\}) = (1, 2)$.

(b) If $\gimel_\triangleright = 3, 5, \dots 2\mathrm{u}_\triangleright - 1$, then $r(x_{\gimel_\triangleright+1}|\{y_1, y_{\gimel_\triangleright}\}) = r(x_{\gimel_\triangleright+2}|\{y_1, y_{\gimel_\triangleright}\})$.

(c) If $\gimel_\triangleright = 4, 6, \dots 2\mathrm{u}_\triangleright$, then $r(x_2|\{y_1, y_{\gimel_\triangleright}\}) = \left(2, \dfrac{\gimel_\triangleright}{2}\right) = r(x_3|\{y_1, y_{\gimel_\triangleright}\})$.

Case ii: When one vertex belong to the outer cycle and second belong to the inner cycle, then again we can take y_1. If the second vertex is $x_{\gimel_\triangleright}$, then we will find at least two element with the same representation leading to a contraction.

(a) If $\gimel_\triangleright = 1$, then $r(y_2|\{y_1, x_{\gimel_\triangleright}\}) = r(y_{\eth_\triangleright}|\{y_1, x_{\gimel_\triangleright}\})$.

(b) If $\gimel_\triangleright = 2$, then $r(y_3|\{y_1, x_{\gimel_\triangleright}\}) = r(x_{\eth_\triangleright-2}|\{y_1, x_{\gimel_\triangleright}\}) = r(y_{\eth_\triangleright-1}|\{y_1, x_{\gimel_\triangleright}\}$.

(c) If $\gimel_\triangleright = 3, 5, \dots 2\mathrm{u}_\triangleright - 1$, then $r(x_2|\{y_1, x_{\gimel_\triangleright}\}) = r(y_{\eth_\triangleright-1}|\{y_1, x_{\gimel_\triangleright}\})$ and for $\gimel_\triangleright = 2\mathrm{u}_\triangleright + 1$, $r(x_3|\{y_1, x_{\gimel_\triangleright}\}) = r(x_{\eth_\triangleright-1}|\{y_1, x_{\gimel_\triangleright}\})$, $r(x_5|\{y_1, x_{\gimel_\triangleright}\}) = r(x_{\eth_\triangleright-3}|\{y_1, x_{\gimel_\triangleright}\})$.

(d) If $\gimel_\triangleright = 4, 6, \dots, 2\mathrm{u}_\triangleright$, then $r(y_2|\{x_1, x_{\gimel_\triangleright}\}) = r(y_{\eth_\triangleright}|\{y_1, x_{\gimel_\triangleright}\})$.

Case iii: When both the vertex belong to the inner cycle, then we can take x_1. If the second vertex is $x_{\gimel_\triangleright}$, then we will find at least two element with the same representation leading to a contraction.

(a) If $\gimel_\triangleright = 2$, then $r(y_3|\{x_1, y_{\gimel_\triangleright}\}) = r(y_{\eth_\triangleright-1}|\{x_1, x_{\gimel_\triangleright}\})$.

(b) If $\gimel_\triangleright = 3, 5, \dots, 2\mathrm{u}_\triangleright - 1$, then $r(y_{\gimel_\triangleright+1}|\{x_1, x_{\gimel_\triangleright}\}) = r(y_{\gimel_\triangleright+2}|\{y_1, x_{\gimel_\triangleright}\})$ and for $\gimel_\triangleright = 2\mathrm{u}_\triangleright + 1$, $r(x_{2\mathrm{u}_\triangleright}|\{x_1, x_{\gimel_\triangleright}\}) = r(x_{2\mathrm{u}_\triangleright+2}|\{x_1, x_{\gimel_\triangleright}\})$.

(c) If $\gimel_\triangleright = 4, 6, \dots, 2\mathrm{u}_\triangleright$, then $r(y_{\gimel_\triangleright-2}|\{x_1, x_{\gimel_\triangleright}\}) = r(y_{\gimel_\triangleright-1}|\{x_1, x_{\gimel_\triangleright}\})$. All above cases suggest that, $dim(\mathrm{P}(\eth_\triangleright, 2).e) \geq 3$. Which means that, $dim(\mathrm{P}(\eth_\triangleright, 2).e) = 3$. when $\eth_\triangleright \equiv 0(\mod 4)$.

Case 2: $\eth_\triangleright \equiv 2(\mod 4)$

We take $\eth_\triangleright = 4\mathrm{u}_\triangleright + 2$ with $\mathrm{u}_\triangleright \geq 1$ and $\mathrm{u}_\triangleright \in \mathbb{Z}^+$. Again, in this case, we will show that $\{x_1, x_2, x_3\}$ resolves $V(\mathrm{P}(\eth_\triangleright, 2).e)$. Again, x_1 and x_2 will resolve the inner cycle vertices and they will also resolve the outer cycle vertices and the inner cycles. To show that $\{x_1, x_2, x_3\}$ resolves vertices of $V(\mathrm{P}(\eth_\triangleright, 2).e)$, we give vertices representation in $V(\mathrm{P}(\eth_\triangleright, 2).e)$ as regards $\{x_1, x_2\}$. Outer cycle vertices representation are $(y_1|\Re) = (1, 2)$ and $(y_2|\Re) = (2, 1)$,

$$r(y_{2\gimel_\triangleright}|\Re) = \begin{cases} (\gimel_\triangleright + 1, \gimel_\triangleright); & 2 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright + 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 3, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \mathrm{u}_\triangleright + 2 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 3, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 2); & \gimel_\triangleright = 2\mathrm{u}_\triangleright + 1, \end{cases}$$

and

$$r(y_{2\gimel_\triangleright+1}|\Re) = \begin{cases} (\gimel_\triangleright + 1, \gimel_\triangleright + 1); & 1 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 2, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \mathrm{u}_\triangleright + 1 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright - 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 2, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 2); & \gimel_\triangleright = 2\mathrm{u}_\triangleright. \end{cases}$$

And, in the inner cycles,

$$r(x_{2\gimel_\triangleright}|\Re) = \begin{cases} (\gimel_\triangleright + 2, \gimel_\triangleright - 1); & 2 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright + 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 3, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 2); & \mathrm{u}_\triangleright + 2 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright, \end{cases}$$

and

$$r(x_{2\gimel_\triangleright+1}|\Re) = \begin{cases} (\gimel_\triangleright, \gimel_\triangleright + 2); & 1 \leq \gimel_\triangleright \leq \mathrm{u}_\triangleright, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 1, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 4); & \mathrm{u}_\triangleright + 1 \leq \gimel_\triangleright \leq 2\mathrm{u}_\triangleright - 1, \\ (2\mathrm{u}_\triangleright - \gimel_\triangleright + 1, 2\mathrm{u}_\triangleright - \gimel_\triangleright + 3); & \gimel_\triangleright = 2\mathrm{u}_\triangleright. \end{cases}$$

Again, in this case, inner cycle vertices give no common representation. Inner cycles and outer cycle have no same represented vertices. But outer cycle have $r(y_{3+\daleth_{\triangleright}}|\Re) = r(y_{\eth_{\triangleright}-\daleth_{\triangleright}}|\Re)$ for $\daleth_{\triangleright} = 2, 4, \ldots 2\mathrm{u}_{\triangleright} - 2$ and $r(y_3|\Re) = r(y_{\eth_{\triangleright}-1}|\Re)$. Vertex x_3 resolve the same represented vertices as $d(x_3, y_3) = \left\lfloor \frac{3}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-1}) = \left\lfloor \frac{3}{2} \right\rfloor + 2$ and $d(x_3, y_{3+\daleth_{\triangleright}}) = \left\lfloor \frac{3+\daleth_{\triangleright}}{2} \right\rfloor \neq d(x_3, y_{\eth_{\triangleright}-\daleth_{\triangleright}}) = \left\lfloor \frac{3+\daleth_{\triangleright}}{2} \right\rfloor + 2$ for $\daleth_{\triangleright} = 2, 4, \ldots, 2\mathrm{u}_{\triangleright} - 2$. This suggest that $\{x_1, x_2, x_3\}$ resolve vertices of $\mathrm{P}(\eth_{\triangleright}, 2).e)$, which means $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$ when $\eth_{\triangleright} \equiv 2(\mod 4)$.

On the other hand, from Case i, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$. Hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$ for $\eth_{\triangleright} \equiv 2(\mod 4)$.

Case 3: $\eth_{\triangleright} \equiv 1(\mod 4)$
We take $\eth_{\triangleright} = 4\mathrm{u}_{\triangleright} + 1$, with $\mathrm{u}_{\triangleright} \geq 1$ and $\mathrm{u}_{\triangleright} \in \mathbb{Z}^+$. We can see that RS is $\{x_1, x_2, y_3\}$ for standard Petersen graph $(\mathrm{P}(5, 2).e)$. $W = \{x_1, x_2, y_4\}$ resolve all vertices in $(\mathrm{P}(9, 2).e)$ as the represented vertices are $r(y_1|W) = (1, 2, 3)$ and $r(y_2|W) = (2, 1, 2)$ and $r(y_3|W) = (2, 2, 1)$ and $r(y_4|W) = (3, 2, 0)$ and $r(y_5|W) = (3, 3, 1)$ and $r(y_6|W) = (3, 3, 2)$ and $r(y_7|W) = (3, 3, 3)$ and $r(y_8|W) = (2, 2, 4)$ and $r(y_9|W) = (2, 1, 3)$ and $r(x_1|W) = (0, 3, 3)$ and also $r(x_2|W) = (3, 0, 2)$ and $r(x_3|W) = (1, 3, 2)$ and $r(x_4|W) = (4, 1, 1)$ and $r(x_5|W) = (2, 4, 1)$ and also $r(x_6|W) = (2, 2, 2)$ and $r(x_7|W) = (3, 2, 3)$ and $r(x_8|W) = (1, 4, 4)$.

For $\eth_{\triangleright} \geq 13$, we will show that $\{x_1, x_2, y_{2\mathrm{u}_{\triangleright}-1}\}$ resolve vertices of $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ where, $\eth_{\triangleright} \equiv 1(\mod 4)$. For this, first we give vertices representation with respect to $\{x_1, x_2\}$. Outer cycle vertices representation are, $r(y_1|\Re) = (1, 2)$ and $r(y_2|\Re) = (2, 1)$,

$$r(y_{2\daleth_{\triangleright}}|\Re) = \begin{cases} (\daleth_{\triangleright}+1, \daleth_{\triangleright}); & 2 \leq \daleth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (\daleth_{\triangleright}, \daleth_{\triangleright}); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright} + 1, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 3); & \mathrm{u}_{\triangleright} + 2 \leq \daleth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2); & \daleth_{\triangleright} = 2\mathrm{u}_{\triangleright}, \end{cases}$$

$$r(y_{2\daleth_{\triangleright}+1}|\Re) = \begin{cases} (\daleth_{\triangleright}+1, \daleth_{\triangleright}+1); & 1 \leq \daleth_{\triangleright} \leq \mathrm{u}_{\triangleright}, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2); & \mathrm{u}_{\triangleright} + 1 \leq \daleth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1); & \daleth_{\triangleright} = 2\mathrm{u}_{\triangleright}. \end{cases}$$

Now in the inner cycle,

$$r(x_{2\daleth_{\triangleright}}|\Re) = \begin{cases} (\daleth_{\triangleright}+2, \daleth_{\triangleright}-1); & 2 \leq \daleth_{\triangleright} \leq \mathrm{u}_{\triangleright} - 1, \\ (\daleth_{\triangleright}+1, \daleth_{\triangleright}-1); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright}, \\ (\daleth_{\triangleright}-1, \daleth_{\triangleright}-1) & \daleth_{\triangleright} = \mathrm{u}_{\triangleright} + 1, \\ (\daleth_{\triangleright}-3, \daleth_{\triangleright}-1); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright} + 2, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 4); & \mathrm{u}_{\triangleright} + 3 \leq \daleth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 1, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 3); & \daleth_{\triangleright} = 2\mathrm{u}_{\triangleright}. \end{cases}$$

and

$$r(x_{2\daleth_{\triangleright}+1}|\Re) = \begin{cases} (\daleth_{\triangleright}, \daleth_{\triangleright}+2); & 1 \leq \daleth_{\triangleright} \leq \mathrm{u}_{\triangleright} - 1, \\ (\daleth_{\triangleright}, \daleth_{\triangleright}+1); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright}, \\ (\daleth_{\triangleright}, \daleth_{\triangleright}-1); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright} + 1, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1); & \daleth_{\triangleright} = \mathrm{u}_{\triangleright} + 2, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 3, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1); & \mathrm{u}_{\triangleright} + 3 \leq \daleth_{\triangleright} \leq 2\mathrm{u}_{\triangleright} - 2, \\ (2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 2, 2\mathrm{u}_{\triangleright} - \daleth_{\triangleright} + 1); & \daleth_{\triangleright} = 2\mathrm{u}_{\triangleright} - 1. \end{cases}$$

Note that, $\{x_1, x_2\}$ resolves all but the following vertices. y_3 and $y_{\eth_{\triangleright}-1}$, $y_{2\mathrm{u}_{\triangleright}-1}$ and $y_{2\mathrm{u}_{\triangleright}+5}$ and $x_{2\mathrm{u}_{\triangleright}+2}$, $y_{2\mathrm{u}_{\triangleright}+1}$ and $y_{2\mathrm{u}_{\triangleright}+2}$ and also $y_{2\mathrm{u}_{\triangleright}+3}$, $x_{2\mathrm{u}_{\triangleright}-1}$ and $x_{2\mathrm{u}_{\triangleright}+4}$. We can see that, outer cycle vertices which have same representation are at dissimilar distances from $y_{2u_{\triangleright}-1}$. That is $d(y_{2u_{\triangleright}-1}, y_{2u_{\triangleright}+5}) = 5$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}+2}) = 3$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}-1}) = 1$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}+4}) = 4$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}}) = 2$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}+5}) = 4$, $d(y_{2u_{\triangleright}-1}, y_{2u_{\triangleright}}) = 1$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}+3}) = 3$, $d(y_{2u_{\triangleright}-1}, x_{2u_{\triangleright}+1}) = 2$, $d(y_{2u_{\triangleright}-1}, y_{2u_{\triangleright}+4}) = 5$. The above discussion suggests that $\{x_1, x_2, y_{2u_{\triangleright}-1}\}$ is a RS for $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$ when $\eth_{\triangleright} \equiv 1(\mod 4)$ and $\eth_{\triangleright} \geq 13$. Hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \leq 3$ for $\eth_{\triangleright} \equiv 1(\mod 4)$.

Arguments show that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$ are analogous with Case 1 hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$ even for $\eth_{\triangleright} \equiv 3(\mod 4)$.

Case 4: $\eth_{\triangleright} \equiv 3(\mod 4)$

We write $\eth_{\triangleright} = 4u_{\triangleright} + 3$ with $u_{\triangleright} \geq 1$ and $u_{\triangleright} \in \mathbb{Z}^+$. It is not tough to observe that $\{x_1, x_2, y_3\}$ for $V(\mathrm{P}(7,2).e)$ is a RS. As $\eth_{\triangleright} \equiv 3(\mod 4)$ and also $\eth_{\triangleright} \geq 11$, we will show that $\{x_1, x_2, y_{2u_{\triangleright}+1}\}$ resolves $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$. We give representation of the vertices with regards as $\{x_1, x_2\}$. Outer cycle vertices representation are $r(y_1|\Re) = (1,2)$ and also $r(y_2|\Re) = (2,1)$,

$$r(y_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+3, 2u_{\triangleright}-\gimel_{\triangleright}+4); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}, \\ (2u_{\triangleright}-\gimel_{\triangleright}+3, 2u_{\triangleright}-\gimel_{\triangleright}+3); & \gimel_{\triangleright} = 2u_{\triangleright}+1, \end{cases}$$

and

$$r(y_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}+1); & 1 \leq \gimel_{\triangleright} \leq u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+3, 2u_{\triangleright}-\gimel_{\triangleright}+3); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}, \\ (2u_{\triangleright}-\gimel_{\triangleright}+3, 2u_{\triangleright}-\gimel_{\triangleright}+2); & \gimel_{\triangleright} = 2u_{\triangleright}+1. \end{cases}$$

Now, inner cycle have,

$$r(x_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}+2, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (\gimel_{\triangleright}-2, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}+2, \\ (2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+5); & u_{\triangleright}+3 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}, \\ (2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+4); & \gimel_{\triangleright} = 2u_{\triangleright}+1, \end{cases}$$

$$r(x_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}+2); & 1 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+3, 2u_{\triangleright}-\gimel_{\triangleright}+2); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}. \end{cases}$$

Again in this case, $\{x_1, x_2\}$ resolve all the vertices in $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ but the following. $y_{3+\gimel_{\triangleright}}$ and $y_{\eth_{\triangleright}-\gimel_{\triangleright}}$ for $\gimel_{\triangleright} = 0, 2, \ldots 2u_{\triangleright} - 4$, $y_{2u_{\triangleright}}$ and $x_{2u_{\triangleright}+2}$, $y_{2u_{\triangleright}+1}$ and $y_{2u_{\triangleright}+5}$ and $x_{2u_{\triangleright}+3}$. We see that the vertices in the outer cycle with the same representation are located at various distances from $y_{2u_{\triangleright}+1}$. That is $d(y_{2u_{\triangleright}+1}, y_{2u_{\triangleright}}) = 1$, $d(y_{2u_{\triangleright}+1}, y_{2u_{\triangleright}+5}) = 4$, $d(y_{2u_{\triangleright}+1}, x_{2u_{\triangleright}+3}) = 2$, $d(y_{2u_{\triangleright}+1}, x_{2u_{\triangleright}+4}) = 3$, $d(y_{2u_{\triangleright}+1}, y_{2u_{\triangleright}+6}) = 5$. Hence, RS $\{x_1, x_2, x_{2u_{\triangleright}+1}\}$ for $V(\mathrm{P}(\eth_{\triangleright}, 2).e)$ when $\eth_{\triangleright} \equiv 3(\mod 4)$ and like the same above discussed cases, arguments similar to Case 1 suggest that, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) \geq 3$, hence, $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = 3$ for $\eth_{\triangleright} \equiv 3(\mod 4)$.

We see that only three vertices suffices to resolve $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ all vertices for any value of $\eth_{\triangleright} \geq 5$ which express that, generalized Petersen graph $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ create a collection of graphs which has constant MD. Since $dim(\mathrm{P}(\eth_{\triangleright}, 2)) = 3$, this shows that $dim(\mathrm{P}(\eth_{\triangleright}, 2)) = dim(\mathrm{P}(\eth_{\triangleright}, 2).e)$.

In Figure 4, (a) Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2)$ and (b) Edge contracted Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2).e$ are discussed.

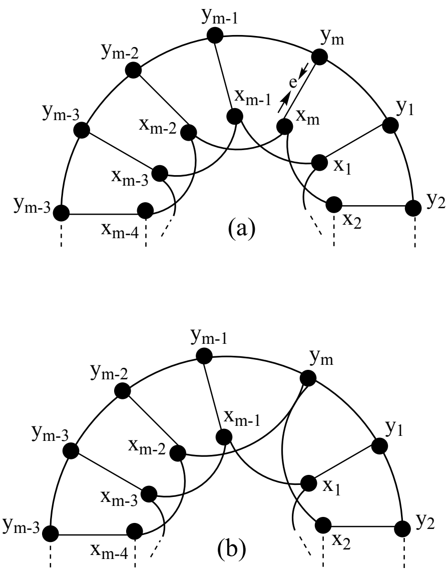

Figure 4: (a) Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2)$. (b) Edge contracted Petersen graph $\mathrm{P}(\eth_{\triangleright}, 2).e$.

□

Theorem 3.3. *Let* $\mathrm{P}(\eth_{\triangleright}, 2)$ *be the generalized Petersen graph and* $(\mathrm{P}(\eth_{\triangleright}, 2).e)$ *be the inner EC of generalized Petersen graph, then,* $dim(\mathrm{P}(\eth_{\triangleright}, 2).e) = dim(\mathrm{P}(\eth_{\triangleright}, 2))$ *for* $\eth_{\triangleright} \geq 5$.

4 Metric Dimension of Edge Contracted Harary Graph

Harary graph $H_{n,\eth_{\triangleright}}$ is an n-regular graph having vertex set $V(H_{n,\eth_{\triangleright}}) = \{v_1, v_2, \dots, v_{\eth_{\triangleright}}\}$ with smallest possible number of edges is $\left[\dfrac{mn}{2}\right]$, where $[x]$ is the ceilling function. Here, selecting landmarks wisely is crucial.

Definition 4.1. [13] *For a graph* $G = (V, E)$ *the Harary edge-contracted graph, signified as* G/e*, is molded by:*

1. *Contracting an edge* $e = (u, v) \in E$*, where u and v are distinctive vertices.*
2. *Combining the vertices u and v into a single vertex w.*
3. *Eliminating any self-loops that form from w to itself.*
4. *While preserving all other edges that were incident to u and v, now attaching them to w.*

Theorem 4.1. *Let* $H_{4,\eth_{\triangleright}}$ *be a 4-regular Harary graph with* $\eth_{\triangleright} \geqslant 5$*; then the outer edge contracted MD of* $H_{4,\eth_{\triangleright}}.e$ *is,*

$$\beta(H_{4,\eth_{\triangleright}}.e) = \begin{cases} \beta(H_{4,\eth_{\triangleright}}), & if \quad \eth_{\triangleright} \equiv 0, 3(\mod 4), \\ 3, & if \quad \eth_{\triangleright} \equiv 1(\mod 4), \\ 4, & otherwise. \end{cases}$$

Proof. Suppose $\Re = \{v_1, v_2, v_3\}$ be any arbitrary subset of $V(H_{4,\eth_{\triangleright}}.e)$. We have to express that $\Re$ distinguish all $H_{4,\eth_{\triangleright}}.e$ vertices except when $\eth_{\triangleright} \equiv 2 \ (\mod 4)$. Here, we discuss few cases.

Case 1: If we take $\eth_{\triangleright} \equiv 0 \pmod 4$ i.e., $\eth_{\triangleright} = 4u_{\triangleright},\ u_{\triangleright}(\geq 3) \in \mathbb{Z}^{+}$, then,

$$r(v_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}-1, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (u_{\triangleright}-1, u_{\triangleright}, u_{\triangleright}); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}, 2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}-1, \end{cases}$$

and also,

$$r(v_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}-1, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}, \\ (2u_{\triangleright}-\gimel_{\triangleright}, 2u_{\triangleright}-\gimel_{\triangleright}, 2u_{\triangleright}-\gimel_{\triangleright}+1); & u_{\triangleright}+1 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}-1. \end{cases}$$

We observe that all $H_{4,\eth_{\triangleright}}.e$ vertices have dissimilar representation as regards $\Re$ which give $dim(H_{4,\eth_{\triangleright}}.e) \leq 3$ when $\eth_{\triangleright} \equiv 0 \pmod 4$.

Case 2: If we take $\eth_{\triangleright} \equiv 1 \pmod 4$ i.e., $\eth_{\triangleright} = 4u_{\triangleright}+1,\ u_{\triangleright}(\geq 3) \in \mathbb{Z}^{+}$, then,

$$r(v_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}-1, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (u_{\triangleright}, u_{\triangleright}, u_{\triangleright}); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}, \end{cases}$$

and also,

$$r(v_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}-1, \\ (\gimel_{\triangleright}, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}, \\ (u_{\triangleright}-1, u_{\triangleright}, u_{\triangleright}); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}, 2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}-1. \end{cases}$$

Once more all $H_{4,\eth_{\triangleright}}.e$ vertices have dissimilar representation as regards $\Re$ which give $dim(H_{4,\eth_{\triangleright}}.e) \leq 3$ when $\eth_{\triangleright} \equiv 1 \pmod 4$.

Case 3: If we take $\eth_{\triangleright} \equiv 3 \pmod 4$ i.e., $\eth_{\triangleright} = 4u_{\triangleright}+3,\ u_{\triangleright}(\geq 3) \in \mathbb{Z}^{+}$, then,

$$r(v_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}-1, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+3); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}+1, \end{cases}$$

and also,

$$r(v_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (\gimel_{\triangleright}-1, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+3); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}. \end{cases}$$

Once more all $H_{4,\eth_{\triangleright}}.e$ vertices have dissimilar representation as regards $\Re$ which give $dim(H_{4,\eth_{\triangleright}}.e) \leq 3$ when $\eth_{\triangleright} \equiv 3 \pmod 4$.

Case 4: If we take $\eth_{\triangleright} \equiv 2 \pmod 4$ i.e., $\eth_{\triangleright} = 4u_{\triangleright}+2,\ u_{\triangleright}(\geq 3) \in \mathbb{Z}^{+}$, then,

$$r(v_{2\gimel_{\triangleright}}|\Re) = \begin{cases} (\gimel_{\triangleright}, \gimel_{\triangleright}-1, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (\gimel_{\triangleright}-1, \gimel_{\triangleright}-1, \gimel_{\triangleright}-1); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2, 2u_{\triangleright}-\gimel_{\triangleright}+3); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}, \end{cases}$$

and also,

$$r(v_{2\gimel_{\triangleright}+1}|\Re) = \begin{cases} (\gimel_{\triangleright}+1, \gimel_{\triangleright}, \gimel_{\triangleright}-1); & 2 \leq \gimel_{\triangleright} \leq u_{\triangleright}, \\ (u_{\triangleright}, u_{\triangleright}, u_{\triangleright}); & \gimel_{\triangleright} = u_{\triangleright}+1, \\ (2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+1, 2u_{\triangleright}-\gimel_{\triangleright}+2); & u_{\triangleright}+2 \leq \gimel_{\triangleright} \leq 2u_{\triangleright}. \end{cases}$$

For $\eth_{\triangleright} \equiv 2 \pmod 4$, we observe that $v_{2u_{\triangleright}+2}$ and $v_{2u_{\triangleright}+3}$ have similar representation. To obtained dissimilar representation, we include $v_{2u_{\triangleright}+2}$ to $\Re$. That is $\Re_1 = \{v_1, v_2, v_3, v_{2u_{\triangleright}+2}\}$

distinguish $V(H_{4,\eth_{\triangleright}}.e)$. Hence $dim(H_{4,\eth_{\triangleright}}.e) = 4$ for $\eth_{\triangleright} \equiv 2 \pmod 4$. Conversely we express, $dim(H_{4,\eth_{\triangleright}}.e) \leq 3$ for $\eth_{\triangleright} \equiv 0, 1, 3 \pmod 4$. Assume in contrary that this is not true. We suppose that $dim(H_{4,\eth_{\triangleright}}.e) = 2$ for $\eth_{\triangleright} = 4u_{\triangleright} + p$, where p can be 0, 1 and 3.

Suppose v_1 is one vertex and the second vertex is v_i, then,

(i) If $\daleth_{\triangleright} = 2$, then $r(v_3|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_4|\{v_1, v_{\daleth_{\triangleright}}\})$.

(ii) For $\eth_{\triangleright} \equiv 0 \pmod 4$,

- (a) If $3 \leq \daleth_{\triangleright} \leq 2\wp_{\triangleright} - 1$, then $r(v_{\daleth_{\triangleright}+5}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+6}|\{v_1, v_{\daleth_{\triangleright}}\})$.
- (b) If $\daleth_{\triangleright} = 4$, then $r(v_{\daleth_{\triangleright}+1}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+2}|\{v_1, v_{\daleth_{\triangleright}}\})$.

(iii) For $\eth_{\triangleright} \equiv 1 \pmod 4$,

- (a) If $\daleth_{\triangleright} = 4$, then $r(v_{\daleth_{\triangleright}+3}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+4}|\{v_1, v_{\daleth_{\triangleright}}\})$.
- (b) If $\daleth_{\triangleright} = 6$, then $r(v_{\daleth_{\triangleright}+1}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+2}|\{v_1, v_{\daleth_{\triangleright}}\})$.

(iv) For $\eth_{\triangleright} \equiv 3 \pmod 4$,

- (a) If $\daleth_{\triangleright} = 4$, then $r(v_{\daleth_{\triangleright}+1}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+2}|\{v_1, v_{\daleth_{\triangleright}}\})$, $r(v_{\daleth_{\triangleright}+3}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+4}|\{v_1, v_{\daleth_{\triangleright}}\})$, $r(v_{\daleth_{\triangleright}+5}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+6}|\{v_1, v_{\daleth_{\triangleright}}\})$.
- (b) If $\daleth_{\triangleright} = 6$, then $r(v_{\daleth_{\triangleright}+1}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+2}|\{v_1, v_{\daleth_{\triangleright}}\})$, $r(v_{\daleth_{\triangleright}+3}|\{v_1, v_{\daleth_{\triangleright}}\}) = r(v_{\daleth_{\triangleright}+4}|\{v_1, v_{\daleth_{\triangleright}}\})$, a contradiction again. This gives that $dim(H_{4,\eth_{\triangleright}}.e) = 3$ for $\eth_{\triangleright} \equiv 0, 1, 3 \pmod 4$ whereas $dim(H_{4,\eth_{\triangleright}}.e) = 4$ for $\eth_{\triangleright} \equiv 2 \pmod 4$. We observe that like above discussed regular graphs $H_{4,\eth_{\triangleright}}.e$ vertices are resolved by only three and four vertices for $\eth_{\triangleright} \equiv 0, 1, 3 \pmod 4$ and $\eth_{\triangleright} \equiv 2 \pmod 4$ respectively, which represent that $H_{4,\eth_{\triangleright}}.e$ for $\eth_{\triangleright} \equiv 0, 1, 3 \pmod 4$ is a collection of graphs having constant MD. Since $dim(H_{4,\eth_{\triangleright}}) = 3$ for $\eth_{\triangleright} \equiv 0, 2, 3 \pmod 4$ whereas $dim(H_{4,\eth_{\triangleright}}) = 4$ for $\eth_{\triangleright} \equiv 1 \pmod 4$. That is why outer edge contracted MD of $H_{4,\eth_{\triangleright}}.e$ is,

$$\beta(H_{4,\eth_{\triangleright}}.e) = \begin{cases} \beta(H_{4,\eth_{\triangleright}}), & if \quad \eth_{\triangleright} \equiv 0, 3 \pmod 4, \\ 3, & if \quad \eth_{\triangleright} \equiv 1 \pmod 4, \\ 4, & otherwise. \end{cases}$$

□

In Figure 5, (a) Harary graph $H_{4,\eth_{\triangleright}}$ and (b) Edge contracted Harary graph $H_{4,\eth_{\triangleright}}.e$ are discussed.

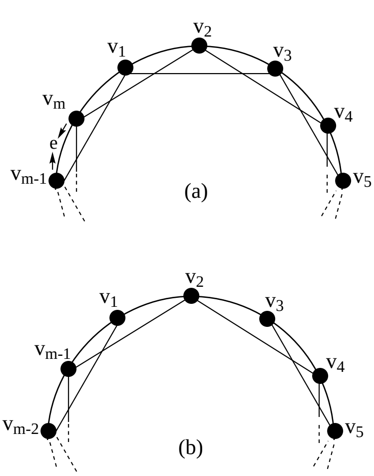

Figure 5: (a) Harary graph $H_{4,\eth_{\triangleright}}$. (b) Edge contracted Harary graph $H_{4,\eth_{\triangleright}}.e$.

Theorem 4.2. *Let $H_{4,\eth_\triangleright}$ be a 4-regular Harary graph with $\eth_\triangleright \geqslant 5$; then the inner edge contracted MD of $H_{4,\eth_\triangleright}.e$ is,*

$$\beta(H_{4,\eth_\triangleright}.e) = \begin{cases} \beta(H_{4,\eth_\triangleright}), & if \quad \eth_\triangleright \equiv 0, 3 \pmod 4), \\ 3, & if \quad \eth_\triangleright \equiv 1 \pmod 4), \\ 4, & otherwise. \end{cases}$$

5 Conclusion

An analysis of the MD in edge contracted regular graphs reveals an intriguing aspect of graph theory. Scholars have learned vital information about the dimensionality and structural properties of graphs by studying the behavior of MD under ECs. In this article, the MD for certain families of graphs have been investigated.

- $dim(A_{\eth_\triangleright}.e) = dim(A_{\eth_\triangleright})$ for $\eth_\triangleright \geqslant 3$.
- $dim(\mathrm{P}(\eth_\triangleright, 2).e) = dim(\mathrm{P}(\eth_\triangleright, 2))$ for $\eth_\triangleright \geq 5$.
- $$\beta(H_{4,\eth_\triangleright}.e) = \begin{cases} \beta(H_{4,\eth_\triangleright}), & if \quad \eth_\triangleright \equiv 0, 3 \pmod 4), \\ 3, & if \quad \eth_\triangleright \equiv 1 \pmod 4), \\ 4, & otherwise. \end{cases}$$

5.1 Limitations of the study

By focusing on certain families such as antiprism, Petersen, and Harary graphs, the conclusions become more generalizable and important. Although these families shed light on symmetrical and regular structures, other graph types, such as irregular or sparse graphs, may display distinct behaviors under EC, which this work more addresses.

5.2 Future work

Exploring other graph families and structures: future research could broaden the scope to include a broader range of graph families, such as trees, bipartite graphs, and scale-free networks, to determine whether the patterns observed with EC and MD are universal or if different structures exhibit distinct behaviors.

Additional invariants: to offer a more complete picture, future research might look at how other invariants (such as diameter, connectedness, and chromatic number) change during EC. This would not only improve our knowledge of EC effects, but it can also reveal interdependencies between MD and the other invariants.

Developing scalable algorithms: as bigger and more complicated networks are explored, inventing algorithms that efficiently compute MDs under EC will become critical. Future research might focus on developing scalable techniques or approximation algorithms to handle large networks more effectively.

Application-oriented studies: applying discoveries to real-world networks, such as biological, social, or transportation networks, would put theoretical ideas to the test in practice. Researchers could, for example, investigate how contracted graph models with specified MDs aid in the optimization of routing, navigation, and resource placement.

Interplay between metric dimension and graph topology: future study might look into how topology-specific properties (such as clustering, assortativity, or degree distribution) interact with MD during EC, perhaps leading to a more general theory that connects graph topology with distance-based metrics.

Future research that addresses these restrictions and expands in these areas can contribute to a thorough knowledge of MD and other invariants in contracted graph models, making this field of study useful for both theoretical breakthroughs and practical applications.

Acknowledgement The authors thank to Universiti Malaysia Terengganu for the support of this research work.

Conflict of Interest The authors declare no conflict of interest.

References

[1] A. S. Alali, S. Ali, N. Hassan, A. M. Mahnashi, Y. Shang & A. Assiry (2023). Algebraic structure graphs over the commutative ring $\mathbb{Z}_m$: Exploring topological indices and entropies using $\mathbb{M}$-polynomials. *Mathematics*, *11*(18), Article ID: 3833. https://doi.org/10.3390/math11183833.

[2] A. Alamer, H. Zafar & M. Javaid (2023). Study of modified prism networks via fractional metric dimension. *AIMS Mathematics*, *8*(5), 10864–10886. https://doi.org/10.3934/math.2023551.

[3] S. Ali, R. M. Falcón & M. K. Mahmood. Local fractional metric dimension of rotationally symmetric planar graphs arisen from planar chorded cycles. arXiv: Combinatorics 2021. https://doi.org/10.48550/arXiv.2105.07808.

[4] D. A. Alrowaili, U. Ahmad, S. Hamed & M. Javaid (2023). Graphs with mixed metric dimension three and related algorithms. *AIMS Mathematics*, *8*(7), 16708–16723. https://doi.org/10.3934/math.2023854.

[5] H. Alshehri, A. Ahmad, Y. Alqahtani & M. Azeem (2022). Vertex metric-based dimension of generalized perimantanes diamondoid structure. *IEEE Access*, *10*, 43320–43326. https://doi.org/10.1109/ACCESS.2022.3169277.

[6] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara & D. R. Wood (2007). On the metric dimension of cartesian products of graphs. *SIAM Journal On Discrete Mathematics*, *21*(2), 423–441. https://doi.org/10.1137/050641867.

[7] G. Chartrand, L. Eroh, M. A. Johnson & O. R. Oellermann (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, *105*(1-3), 99–113. https://doi.org/10.1016/S0166-218X(00)00198-0.

[8] M. Feng & K. Wang (2013). On the metric dimension and fractional metric dimension of the hierarchical product of graphs. *Applicable Analysis and Discrete Mathematics*, *7*(2), 302–313. https://doi.org/10.2298/AADM130521009F.

[9] M. Feng, M. Xu & K. Wang (2013). On the metric dimension of line graphs. *Discrete Applied Mathematics*, *161*(6), 802–805. https://doi.org/10.1016/j.dam.2012.10.018.

[10] K. J. Gowtham, M. N. Husin & M. Siddiqui (2024). Some new bounds on the modified symmetric division deg index. *Malaysian Journal of Mathematical Sciences*, *18*(1), 39–50. https://doi.org/10.47836/mjms.18.1.03.

[11] F. Harary & R. A. Melter (1976). On the metric dimension of a graph. *Ars Combinatoria*, *2*, 191–195.

[12] H. Iswadi, E. T. Baskoro & R. Simanjuntak (2011). On the metric dimension of corona product of graphs. *Far East Journal of Mathematical Sciences*, *52*(2), 155–170.

[13] Q. Liu, X. Huang & Z. Zhang (2011). Restricted edge connectivity of Harary graphs. In *Combinatorial Optimization and Applications: 5th International Conference, COCOA 2011*, pp. 113–125. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-22616-8_10.

[14] R. A. Melter & I. Tomescu (1984). Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, *25*(1), 113–121. https://doi.org/10.1016/0734-189X(84)90051-3.

[15] L. Saha, R. Lama & K. Tiwary (2022). Metric dimension vulnerability in trees under edge contraction. *Palestine Journal of Mathematics*, *11*(3), 249–254.

[16] S. W. Saputro, E. T. Baskoro, A. N. M. Salman & D. Suprijanto (2009). The metric dimensions of a complete n-partite graph and its cartesian product with a path. *Journal of Combinatorial Mathematics and Combinatorial Computing*, *71*, 283–293.

[17] S. W. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. T. Baskoro, A. Salman & M. Bača (2013). The metric dimension of the lexicographic product of graphs. *Discrete mathematics*, *313*(9), 1045–1051. https://doi.org/10.1016/j.disc.2013.01.021.

[18] S. J. Seo & P. J. Slater (2017). Open-independent, open-locating-dominating sets. *Electronic Journal of Graph Theory & Applications*, *5*(2), 179–193. https://doi.org/10.5614/ejgta.2017.5.2.2.

[19] H. M. A. Siddiqui, K. Mazhar, M. K. Siddiqui & M. F. Nadeem (2025). On fault-tolerant metric dimension of supramolecular networks. *Discrete Mathematics, Algorithms & Applications*, *17*(1), Article ID: 2450002. https://doi.org/10.1142/S1793830924500022.

[20] P. J. Slater (1988). Dominating and reference sets in a graph. *Journal of Mathematical and Physical Sciences*, *22*(4), 445–455.

[21] B. Sooryanarayana, S. Kunikullaya & N. N. Swamy (2019). Metric dimension of generalized wheels. *Arab Journal of Mathematical Sciences*, *25*(2), 131–144. https://doi.org/10.1016/j.ajmsc.2019.04.002.

[22] M. Sun (2023). PP-GNN: Pretraining position-aware graph neural networks with the NP-hard metric dimension problem. *Neurocomputing*, *561*, Article ID: 126848. https://doi.org/10.1016/j.neucom.2023.126848.

[23] I. G. Yero, D. Kuziak & J. A. Rodríguez-Velázquez (2011). On the metric dimension of corona product graphs. *Computers & Mathematics with Applications*, *61*(9), 2793–2798. https://doi.org/10.1016/j.camwa.2011.03.046.